Contact Structures of Partial Differential Equations

Pieter Thijs Eendebak

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Contact Structures of Partial Differential Equations

Contact Structuren voor Partiële Differentiaalvergelijkingen

(met een samenvatting in het Nederlands)

Proefschrift

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Pieter Thijs Eendebak

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Promotor: prof.dr. J.J. Duistermaat

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Contents

Introduction

In this dissertation we study a new method for solving partial differential equations. The method is a generalization of the symmetry methods developed by Lie and the method of Darboux. We call this method the projection method. Sometimes we also speak of (vector) pseudosymmetries.

We study projections in the context of two special classes of partial differential equations. These two classes are the first order systems in two independent and two dependent variables and the second order scalar equations in two independent variables. The equation manifold for a first order system has dimension 6 and the equation manifold for a second order equation has dimension 7. For both types of equations there is a canonical rank 4 distribution that (in the case of contact geometry) completely describes the equation. The analysis of this rank 4 distribution, or the dual Pfaffian system, is called the geometric theory of partial differential equations and is another main subject of this dissertation.

The analysis of the geometry of partial differential equations was started in the 19th century by mathematicians including Monge, Darboux, Lie, Goursat, Cartan and Vessiot. After a period of relatively few developments, this branch of mathematics has seen a revival in recent years with contributions from Bryant et al. [14], Gardner and Kamran [38], Juráš [44], Stormark [64], Vassiliou [66] and many others. The author's thesis advisor prof.dr. J.J. Duistermaat was also interested in the geometric approach to partial differential equations and his study led to the article [25] in which he analyzes the minimal surface equation. The method used to find solutions for the minimal surface equation in the article is a special case of the real method of Darboux in the elliptic case. The analysis of the minimal surface equation was done by taking the quotient of the system by the translation symmetries. The projection onto the quotient manifold is an example of the projections studied in this dissertation. In suitable local coordinates this method corresponds to the Weierstrass representation.

Duistermaat recognized that also for other types of equations such projections could exist and could be used in solving the equations. While for the minimal surface equation the projection is generated by symmetries, projections not generated by symmetries also exist. These projections can still be used to solve partial differential equations.

This dissertation has three main components. The first is the structure theory for first order systems and second order equations. This is developed in chapters 4, 5 and 6. The second component is the theory of Darboux integrability. In Chapter 8 we introduce Darboux integrability and give a complete classification of the first order systems that are Darboux integrable on the first order jets. In Chapter 10 we give a geometric construction of the Lie algebras associated to Darboux integrable equations by Vessiot [69, 70]. The last component is the projection method that appears in various forms throughout the thesis. The projection can be either tangent to or transversal to the contact distribution. The first case leads to projections to a base manifold, these are studied in Chapter 7. The transversal projections are studied in Chapter 9. Chapter 11 contains a summary of the different projection methods.

Assumptions. We assume the reader is familiar with the basic concepts of differential geometry, such as Lie groups, vector fields, differential forms, tensors and representations. We also assume that the reader has some basic notion of a partial differential equation. Finally we assume the reader is familiar with exterior differential systems, integral elements, the Cartan-Kähler theorem and the method of equivalence. An introduction to these topics can be found in Ivey and Landsberg [43]. A more detailed analysis of exterior differential systems and the Cartan-Kähler theorem can be found in Bryant et al. [13]. The method of equivalence is described in more detail in Gardner [37]. In Chapter 1 we repeat the basic definitions for the objects used in this dissertation and give more precise references.

Notation and conventions. The end of a proof will be indicated by the symbol $\square$. The end of a definition, example or remark by the symbol $\oslash$. In the text we will use the Einstein summation convention.

For convenience we assume that all functions and structures defined in this dissertation are smooth (C^∞). Whenever we apply the Cartan-Kähler theorem, we assume the structures involved are analytic. We will also assume that (unless stated otherwise) all vector bundles (distributions, Pfaffian systems, etc.) are locally of constant rank. This means we will almost always avoid any singularities. The condition seems quite restrictive, but often we can restrict to open subsets away from the singularities. The constant-rank assumption ensures that we can switch between vector fields and pointwise constructions. Most constructions are local constructions. If it is necessary (for obvious reasons) to restrict to a small neighborhood to carry out constructions or computations, then we will not always explicitly mention this.

Computations. Almost all of the computations in this dissertation have either been performed with or checked with computer algebra systems. In particular MAPLE [71] and MATHEMATICA [75] have been used extensively. For calculations with vector fields and differential forms the packages JETS [7] by Mohamed Barakat and Gehrt Hartjen, the package VESSIOT [3] by Ian Anderson and the MAPLE package DIFFORMS have been used. Some special purpose packages have been written by the author and are available on the author's homepage [32].

Dissertation. This dissertation is a continuation of the work in my Master's thesis [31]. One of the main questions we had when I finished my Master's thesis, was whether we could generalize the projection method for the minimal surface equation [25] to a more general class of equations. When I started working on this problem in 2002, I used computer algebra systems (MAPLE) to see if there are any finite order obstructions to the existence of a projection. It soon turned out that this is a computationally very difficult problem and the approach taken (brute force calculations) did not give much intuition for the problem. At that time I also met Mohamed Barakat from whom I learned his computer algebra system JETS [7]. I also started reading the book on exterior differential systems by Bryant et al. [13].

In 2003 I started working on symmetries of partial differential equations [58, 59] and on Darboux integrability [44, 64, 66]. Both topics are closely related to the projection method: every system with enough symmetries allows a projection and the Darboux integrable equations provide examples of projections that are not generated by symmetries.

One of the turning points in my research was the introduction to the work of McKay [51, 52]. McKay developed a structure theory for first order systems and one of the questions he asked at the end of his thesis is whether a second order equation can be projected to a first order system or not. His question is a question about possible projections! Inspired by this question I started reading his work and learned very soon that first order systems and second order equations have a very similar structure. I also learned that many of the structures Duistermaat and I had found using distributions for second order equations corresponded directly to structures for first order systems McKay had found using differential forms and the method of equivalence.

In the first half of 2005 Barakat, Duistermaat and I tried to apply the theory of jet groupoids (developed by Pommaret [61]) to our geometric structures. The theory works better for first order systems than for second order equations. For first order systems the theory allowed us to find the number of continuous invariants of a first order system at a given order. In the summer of 2005 I also visited Florida, USA. Together with Robert Bryant I developed a method for calculating pseudosymmetries of second order equations (these are special examples of projections).

In the remainder of 2005 and 2006 we used the structure theory we have developed to study classifications of Darboux integrable equations, generalizations of Darboux integrability, pseudosymmetries and other topics related to our research.

Chapter 1

General theory

1.1 Notation

1.1.1 Differential ideals

The space of all k-forms on a smooth manifold M will be denoted by $\Omega^k(M)$; the algebra of smooth differential forms on M is denoted by $\Omega^*(M)$. The *algebraic ideal* generated by a set of differential forms α^j will be denoted by $\{\alpha^j\}_{\mathrm{alg}}$; the *differential ideal* (see Section 1.2.2) generated by the same set of elements is denoted by $\{\alpha^j\}_{\mathrm{diff}}$.

If we have a set of 1-forms α^j we can form both the algebraic ideal in the algebra $\Omega^*(M)$ and the ideal of 1-forms in the module $\Omega^1(M)$. The $C^\infty(M)$-module generated by a set of differential forms as a module in $\Omega^1(M)$ will be denoted by $\mathrm{span}(\alpha^1, \ldots, \alpha^j)$. This is both an ideal in the $C^\infty(M)$-module $\Omega^1(M)$ and a subbundle of T^*M.

We denote the interior product of a vector X with a k-form ω by $X \lrcorner\, \omega$.

1.1.2 Dual vector fields

The space of vector fields on a manifold M is denoted by $\mathscr{X}(M)$. Given a basis of differential forms θ^j, $j = 1, \ldots, n$, we define the dual vector fields as the vector fields X_j that satisfy $\theta^i(X_j) = \delta^i_j$. We will write $\partial/\partial\theta^j$ or ∂_{θ^j} for the vector field X^j dual to θ^j. Note that the vector field ∂_{θ^1} cannot be determined from the differential form θ^1 alone. We need the full basis θ^j, $j = 1, \ldots, n$ in order to determine ∂_{θ^1}.

If we have introduced local coordinates $x^1, \ldots, x^n$ on a manifold, then we have a natural basis of differential forms $\theta^j = \mathrm{d}x^j$, $j = 1, \ldots, n$. We will then write ∂_{x^j} for the vector field $\partial_{\theta^j} = \partial_{\mathrm{d}x^j}$.

Example 1.1.1. Consider the basis of differential forms on $\mathbb{R}^3$ given by $\theta^1 = \mathrm{d}x$, $\theta^2 = \mathrm{d}y + 4\mathrm{d}x$, $\theta^3 = \mathrm{d}z + x\mathrm{d}x$. The dual vector fields are given by

$$\partial_{\theta^1} = \partial_x - 4\partial_y - x\partial_z, \quad \partial_{\theta^2} = \partial_y, \quad \partial_{\theta^3} = \partial_z. \qquad \oslash$$

1.1.3 Vector bundles

Let E, F and G be vector bundles over the base manifold M. We will write $F \times_M G$ for the *fibered product* of F and G. This is the vector bundle over M for which the fiber over $x \in M$ is $(F \times_M G)_x = F_x \times G_x$. If F and G are vector subbundles of E we use the notation $F + G$ for the vector subbundle of E defined by $(F + G)_x = \{ X + Y \in E_x \mid X \in F_x, Y \in G_x \}$. If for every point $x \in M$ the vector space E_x is the direct sum of F_x and G_x, we will write $E = F \oplus G$. This notation should not be confused with the Whitney sum of F and G.

1.1.4 Jet bundles

Let M and N be smooth manifolds. The *jet bundle* of k-jets of functions from M to N will be denoted by $\mathrm{J}^k(M, N)$. The k-jet of a smooth function $\phi : M \to N$ at a point x will be denoted by $j^k_x\phi$. If we have coordinates $x^1, \ldots, x^m$ for M and $y^1, \ldots, y^n$ for N, then we can introduce coordinates x^i, y^a, p^a_i for $\mathrm{J}^1(M, N)$. The 1-jet of a function $\phi : M \to N$ at x is given by $(x^i, y^a, p^a_i) = (x^i, \phi^a(x), (\partial\phi^a/\partial x^i)(x))$. On the second order jet bundle $\mathrm{J}^2(M, N)$ we have coordinates $x^i, y^a, p^a_i, p^a_{ij}$, etc.

On every jet bundle there is a natural ideal of contact forms. In the local coordinates introduced above this ideal is generated by contact forms of the form

$$\theta^a_I = \mathrm{d}p^a_I - p^a_{I,k}\mathrm{d}x^k,$$

with I a multi-index $I = (i_1, \ldots, i_s)$.

Every transformation of the base manifold $M \times N$ can be prolonged to a unique transformation on the jet bundle $\mathrm{J}^k(M, N)$ that preserves the contact ideal. This prolongation is called the induced point transformation. The prolongation might only be defined on a subset. In a similar way we can prolong vector fields on the base manifolds to unique vector fields on the jet bundle that are symmetries of the contact structure on the jet bundle.

We will use the terminology *base transformations* for the transformations of the base manifold, *point transformations* for the induced transformations on the jet bundle and *contact transformation* for the general transformations on the jet bundle that preserve the contact structure. For a more detailed discussion of jet bundles and prolongations see Olver [59, Chapter 4] or Bryant et al. [13, Section I.3].

1.2 Basic geometry

In this section we discuss some basic topics in differential geometry. We will only give definitions and some examples and refer the reader for proofs or more detailed theory to other sources.

1.2.1 Lie groups

Given a Lie group G with identity element e we will denote the Lie algebra as $\mathfrak{g} = T_eG$. The *right translation* and *left translation* by an element $g \in G$ on the Lie group will be denoted by R_g and L_g, respectively. We also write $\mathfrak{g}_L$ and $\mathfrak{g}_R$ for the Lie algebras of left- and right-invariant vector fields, respectively. Given an element $X \in \mathfrak{g}$ there is a unique left-invariant vector field X^L that satisfies $X^L(e) = X$. In formula:

$$X^L(g) = (T_e L_g)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g(\exp tX).$$

Similarly, X^R is the unique right-invariant vector field with $X^R(e) = X$. We will identify the Lie algebra $\mathfrak{g}$ with the space $\mathfrak{g}_L$ of left-invariant vector fields.

Lemma 1.2.1. *The left-invariant and right-invariant vector fields act as derivatives of right- and left multiplications, respectively. Let $\Phi^t = \exp(tX^L)$ be the flow of X^L after time t. Then*

$$\Phi^t = R_{\exp(tX)}.$$

For each Lie group there is a unique right-invariant $\mathfrak{g}$-valued 1-form α_R, the right-invariant *Maurer-Cartan* form. This form is defined as

$$(\alpha_R)_g(X) = (T_g R_{g^{-1}})(X) = (T_e R_g)^{-1}(X).$$

There is also a unique left-invariant Maurer-Cartan form, defined as

$$(\alpha_L)_g(X) = (T_g L_{g^{-1}})(X) = (T_e L_g)^{-1}(X).$$

The Maurer-Cartan forms satisfy the structure equations

$$\mathrm{d}\alpha_L(X, Y) = -[\alpha_L(X), \alpha_L(Y)], \quad \mathrm{d}\alpha_R(X, Y) = [\alpha_R(X), \alpha_R(Y)]. \tag{1.1}$$

Matrix groups

If G is realized as a subgroup of $\mathrm{GL}(n, \mathbb{R})$, then the Maurer-Cartan forms can be written as

$$\alpha_R = (\mathrm{d}g)g^{-1}, \quad \alpha_L = g^{-1}(\mathrm{d}g).$$

Example 1.2.2 (Affine group). The affine group $\mathrm{Aff}(n)$ is the group of affine transformations of $\mathbb{R}^n$. An affine transformation of $\mathbb{R}^n$ is a transformation of the form $x \mapsto Ax + b$ with $A \in \mathrm{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. The affine group is the semi-direct product of $\mathrm{GL}(n, \mathbb{R})$ and $\mathbb{R}^n$. A matrix representation of $\mathrm{Aff}(n)$ is given by the space of $(n + 1) \times (n + 1)$-matrices

$$\begin{pmatrix} A & b \\ 0 & I \end{pmatrix}, \tag{1.2}$$

with $A \in \mathrm{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. The group operation is the usual matrix multiplication.

In the case of $\mathrm{Aff}(1)$ we can use coordinates $a \in \mathbb{R}^*$, $b \in \mathbb{R}$ and the representation

$$g = (a, b) \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}).$$

The left- and right-invariant Maurer-Cartan forms are given by

$$\alpha_L = g^{-1}\mathrm{d}g = \begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^{-1}\mathrm{d}a & a^{-1}\mathrm{d}b \\ 0 & 0 \end{pmatrix},$$
$$\alpha_R = \begin{pmatrix} a^{-1}\mathrm{d}a & a^{-1}\mathrm{d}a + \mathrm{d}b \\ 0 & 0 \end{pmatrix}.$$

A basis for the left-invariant vector fields is

$$X_1 = a\partial_a, \quad X_2 = a\partial_b \tag{1.3}$$

and a basis for the right-invariant vector fields is

$$Y_1 = a\partial_a + b\partial_b, \quad Y_2 = \partial_b. \tag{1.4}$$

⊘

1.2.2 Exterior differential systems

Let M be a smooth manifold. The algebra of differential forms on M is a graded algebra $\Omega^*(M) = \bigoplus_k \Omega^k(M)$. An *ideal* I in $\Omega^*(M)$ is an additive subgroup of $\Omega^*(M)$ that is closed under the wedge product (for $\alpha \in I$ and $\beta \in \Omega^*(M)$ the form $\alpha \wedge \beta$ is in I). An ideal is called *homogeneous* if the ideal is a direct sum $I = \bigoplus_k I^k$ with $I^k \subset \Omega^k(M)$. In this dissertation all ideals are assumed to be homogeneous. A *differential ideal* is a homogeneous ideal $\mathcal{I}$ that is not only closed under addition and the wedge product, but also under exterior differentiation. An *exterior differential system* on M is a differential ideal $\mathcal{I} \subset \Omega^*(M)$. For a more complete introduction to exterior algebras and exterior differential ideals see Bryant et al. [13, pp. 6–18].

Integrable elements and integral manifolds

For a k-form ω and a linear subspace $E \subset T_xM$ we denote by ω_E the restriction $\omega|_{E\times\ldots\times E}$ of ω to E.

Definition 1.2.3 (Integral element). A linear subspace $E \subset T_xM$ is an *integral element* of $\mathcal{I}$ if $\omega_E = 0$ for all $\omega \in \mathcal{I}$. The set of integral elements of dimension k of an exterior differential system $\mathcal{I}$ will be denoted by $V_k(\mathcal{I})$.

We define the *polar space* of a k-dimensional integral element E at x to be

$$H(E) = \{\, X \in T_xM \mid \alpha(X, e_1, \ldots, e_k) = 0 \text{ for all } \omega \in \mathcal{I}^{k+1} \,\}.$$

We define $r(E) = \dim H(E) - k - 1$. This number is called the *extension rank* of E. A *maximal* integral element is an integral element E for which $r(E) = -1$. The maximal integral elements are precisely the integral elements that are not contained in any integral element of larger dimension.

Definition 1.2.4 (Integral manifold). A submanifold S of M is called an *integral manifold* of $\mathcal{I}$ if for the natural inclusion $\iota : S \to M$ we have $\iota^*\omega = 0$ for all $\omega \in \mathcal{I}$. $\ominus$

An integral manifold is called *maximal* if for every point in the manifold the tangent space at that point is a maximal integral element. Definition 1.2.4 can be given for any smooth map from a k-dimensional manifold S to M. This would allow for instance for immersed integral manifolds that can be used in global applications.

A k-form is called *decomposable* if it can be written as a monomial $\omega = \omega^1 \wedge \ldots \wedge \omega^k$ with ω^j, $j = 1, \ldots, k$ all 1-forms.

Definition 1.2.5 (Independence condition). An exterior differential system with *independence condition* on a manifold M is a pair $(\mathcal{I}, \Omega)$ where $\mathcal{I}$ is an exterior differential ideal and Ω a decomposable n-form such that $\Omega_x \notin \mathcal{I}_x$ for all $x \in M$. Two independence forms Ω, Ω' are equivalent if and only if $\Omega' \equiv f\Omega \mod \mathcal{I}$ for some $f \in C^\infty(M)$.

The n-dimensional integral elements of an exterior differential system $\mathcal{I}$ with independence condition Ω are the n-dimensional integral elements E of $\mathcal{I}$ for which $\Omega|_E \neq 0$. $\ominus$

Example 1.2.6 (Independence condition). An independence condition is used often as a transversality condition. For example the graphs of the 1-jets of functions $z(x)$ are submanifolds of $\mathrm{J}^1(\mathbb{R})$. These submanifolds are integral manifolds of the exterior differential system

$$\mathcal{I} = \{\mathrm{d}z - p\mathrm{d}x\}_{\mathrm{diff}}\,.$$

The converse is not true. For example the submanifold S defined by $x = z =$ constant is an integral manifold of $\mathcal{I}$, but S does not correspond (not even locally) to the graph of the 1-jet of a function $z(x)$.

We define the independence condition $\Omega = \mathrm{d}x$. Then it is clear that $\Omega|_S = 0$. The integral manifolds S of $\mathcal{I}$ that satisfy $\Omega|_S \neq 0$ can locally be written as the graphs of 1-jets of functions. $\oslash$

Prolongations

Let $\mathrm{Gr}_n(TM)$ be the Grassmannian of n-planes in TM. This is a bundle $\pi : \mathrm{Gr}_n(TM) \to M$ over the base manifold M. A point in the Grassmannian $\mathrm{Gr}_n(TM)$ is given by a pair (x, E) where $X \in M$ and E is a linear subspace of T_xM. Often we will denote a point in the Grassmannian only by E and write $x = \pi(E)$. For every point (x, E) we define $C_E = (T_{(x,E)}\pi)^{-1}(E) \subset T_{(x,E)}\,\mathrm{Gr}_n(TM)$. We define $I_{(x,E)} = (C_E)^\perp$. An equivalent definition is $I_{(x,E)} = \pi^*(E^\perp)$. The bundle I defines the canonical contact system on $\mathrm{Gr}_n(TM)$.

Every n-dimensional submanifold of TM can be lifted to a unique integral submanifold of the exterior differential system on $\mathrm{Gr}_n(TM)$ generated by I. For a submanifold $U \subset M$ this lift is defined as

$$U \to \mathrm{Gr}_n(TM) : u \mapsto (u, T_uU).$$

This lifting gives a local one-to-one correspondence between submanifolds of M and integral manifolds of $(\mathrm{Gr}_n(TM), I)$ that are transversal to the projection π.

Let $\mathcal{I}$ be an exterior differential system on M. The n-dimensional integral elements of $\mathcal{I}$ form a subset $M^{(1)} = V_n(\mathcal{I}) \subset \mathrm{Gr}_n(TM)$. If the space of integral elements is (locally) a smooth manifold, then we can pull back the contact structure on $\mathrm{Gr}_n(TM)$ to $M^{(1)}$. This defines a new exterior differential system $\mathcal{I}^{(1)}$ on $M^{(1)}$. The pair $(M^{(1)}, \mathcal{I}^{(1)})$ is called the *prolongation* of $(M, \mathcal{I})$. The integral manifolds of $\mathcal{I}$ are locally in one-to-one correspondence with the integral manifolds of $\mathcal{I}^{(1)}$ that are transversal to the projection $M^{(1)} \to M$.

1.2.3 Theory of Pfaffian systems

Definition 1.2.7 (Pfaffian system). Let M be a smooth manifold. A *Pfaffian system I* on M is a subbundle of the cotangent bundle. The *dimension* or *rank* of the Pfaffian system is the rank of I as a vector bundle. Our definition of the rank of a Pfaffian system is different from the Engel half-rank and the Cartan rank of a Pfaffian system (see Bryant et al. [13, p. 45] or Gardner [35, §3]). $\ominus$

Remark 1.2.8. In the 19th century a Pfaffian system was a system of equations of the form

$$\begin{cases} \omega_1 = a_{11}\mathrm{d}x_1 + a_{12}\mathrm{d}x_2 + \ldots + a_{1n}\mathrm{d}x_n = 0, \\ \omega_2 = a_{21}\mathrm{d}x_1 + a_{22}\mathrm{d}x_2 + \ldots + a_{2n}\mathrm{d}x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \omega_s = a_{s1}\mathrm{d}x_1 + a_{s2}\mathrm{d}x_2 + \ldots + a_{sn}\mathrm{d}x_n = 0. \end{cases}$$

The $\mathrm{d}x_j$ are formal expressions that have in modern times the interpretation of differential forms. In most modern texts a Pfaffian system is defined as a subbundle of the cotangent space or even as an algebraic ideal of differential forms. If the system is of constant rank 1, then we can even take the dual distribution as the definition of a Pfaffian system. For most applications the different definitions are equivalent. $\oslash$

Every Pfaffian system defines an exterior differential system generated by the sections of the Pfaffian system. Let θ^j be a basis for I. The corresponding exterior differential ideal $\mathcal{I}$ is generated algebraically by the forms θ^j and $\mathrm{d}\theta^j$. Conversely, the essential information in the exterior differential system is already given by the 1-forms, so we can write $I = \mathcal{I} \cap \Omega^1(M)$ for the Pfaffian system. An independence condition $\Omega = \omega^1 \wedge \ldots \omega^n$ for a Pfaffian system is completely determined by the bundle $J = \mathrm{span}(I, \omega^1, \ldots, \omega^n)$. We have $I \subset J \subset T^*M$ and rank $J/I = n$. The independence condition Ω defines a non-zero section of $\Lambda^n(J/I)$.

Definition 1.2.9. Let I be a Pfaffian system. The exterior derivative induces a map

$$\delta : I \to \Omega^2(M)/\{I\}_{\mathrm{alg}}.$$

The (first) *derived system* of I is defined as $I^{(1)} = \ker \delta$. By induction we define the derived flag as $I^{(j+1)} = (I^{(j)})^{(1)}$. $\ominus$

Definition 1.2.10. Let $\mathcal{I}$ be an exterior differential ideal. We define

$$A(\mathcal{I})_x = \{\, \xi_x \in T_x M \mid \xi_x \lrcorner \mathcal{I}_x \subset \mathcal{I}_x \,\},$$
$$C(\mathcal{I}) = A(\mathcal{I})^{\perp} \subset T^*M. \qquad \ominus$$

The space $C(\mathcal{I})$ is called the *retracting space* or *Cartan system* of the exterior differential ideal. The rank of $C(\mathcal{I})$ at a point x is the *class* of $\mathcal{I}$ at x.

The class of a Pfaffian system I is by definition the class of the exterior differential ideal generated by I. The class is equal to the corank of the Cauchy characteristics of $I^{\perp}$. Let I be a Pfaffian system of rank s. Then I is called *integrable* if $\mathrm{d}I \equiv 0 \mod I$.

Theorem 1.2.11 (Frobenius theorem). *Let I be an integrable Pfaffian system of rank s. Then there are local coordinates $y^1, \ldots, y^n$ such that the Pfaffian system I is generated by the 1-forms* $\mathrm{d}y^1, \ldots, \mathrm{d}y^s$.

Lemma 1.2.12 (Cartan's lemma). *Let V be a finite-dimensional vector space and let $v_1, \ldots, v_k$ be linearly independent elements of V. If*

$$\sum_{i=1}^{k} w_i \wedge v_i = 0$$

for vectors w_i, then there exist scalars h_{ij}, symmetric in i and j, such that $w_i = \sum_j h_{ij} v_j$.

Proof. See Ivey and Landsberg [43, Lemma A.1.9] or Sternberg [63, Theorem 4.4]. There is also a higher-degree version of the lemma in which the w_k are multi-vectors; this version is a consequence of the more general Cartan-Poincaré lemma, see Bryant et al. [13, Proposition 2.1]. $\square$

For Pfaffian systems generated by a single 1-form α there are normal forms. Suppose we have the single equation $\alpha = 0$. We define the rank of the equation to be the integer r for which

$$(\mathrm{d}\alpha)^r \wedge \alpha \neq 0, \quad (\mathrm{d}\alpha)^{r+1} \wedge \alpha = 0.$$

The rank r of the equation is invariant under scaling of the 1-form and is equal to the Engel half-rank of the Pfaffian system $\mathrm{span}(\alpha)$. We also define the integer s by

$$(\mathrm{d}\alpha)^s \neq 0, \quad (\mathrm{d}\alpha)^{s+1} = 0.$$

One quickly sees that either $r = s$ or $r = s + 1$. The theorem of Pfaff gives a normal form for α depending on the invariants r, s.

Theorem 1.2.13 (Pfaff theorem). *Let α be 1-form for which r and s are locally constant. Then α has the normal form*

$$\alpha = y^0 \mathrm{d}y^1 + \ldots + y^{2r} \mathrm{d}y^{2r+1}, \qquad \textit{if } r + 1 = s,$$
$$\alpha = \mathrm{d}y^1 + y^2 \mathrm{d}y^3 + \ldots + y^{2r} \mathrm{d}y^{2r+1}, \quad \textit{if } r = s.$$

Here the y^j are part of a local coordinate system.

Proof. See Bryant et al. [13, Theorem II.3.4]. $\square$

1.2.4 Distributions

The objects dual to (constant-rank) Pfaffian systems are distributions. We give here the basic definitions and reformulate some of the previous results in terms of distributions.

Definition 1.2.14. A *distribution* on a smooth manifold M of rank k is a subbundle of the tangent bundle TM of rank k. $\ominus$

We will also use the term *vector subbundle* of the tangent bundle or vector subbundle instead of the term distribution. The name distribution is more common in the literature, but has the drawback of also having a different meaning as a generalized function. Another name used in the literature is a *vector field system*. The distribution spanned by the vector fields $X_1, \dots, X_n$ is denoted by $\operatorname{span}(X_1, \dots, X_n)$.

Remark 1.2.15. For a distribution $\mathcal{V}$ and a vector field X we say that X is contained in $\mathcal{V}$ and write $X \subset \mathcal{V}$ if $X_m \in \mathcal{V}_m$ for all points m. This corresponds to saying that X is contained in $\mathcal{V}$ pointwise. The notation $X \subset \mathcal{V}$ is quite natural if we define a vector field as a section $X : M \to TM$ and identify X with its image $X(M)$. If X is not contained in $\mathcal{V}$ this means that there exists a point m such that $X_m \notin \mathcal{V}_m$. This does not imply that $X_m \notin \mathcal{V}_m$ for all points m. We will say that X is *pointwise not contained* in $\mathcal{V}$ if the stronger statement, that $X_m \notin \mathcal{V}_m$ for all m, holds.

If we write $X \in \mathcal{V}$, then usually X is a vector with $X \in \mathcal{V}_m$ for some point m. $\oslash$

Let I be a constant-rank Pfaffian system. The distribution $\mathcal{V}$ dual to I is defined as

$$\mathcal{V}_x = \{ X \in T_x M \mid \theta(X) = 0, \theta \in I \}.$$

We will denote the dual distribution by $I^{\perp}$.

We say a linear subspace $E \subset TM$ is an *integral element* of $\mathcal{V}$ if E is an integral element of the dual Pfaffian system. We denote the k-dimensional integral elements of $\mathcal{V}$ by $V_k(\mathcal{V})$.

Definition 1.2.16. Let $\mathcal{V}$ be a distribution. The distribution spanned by all smooth vector fields of the form $[X, Y]$ for $X, Y \subset \mathcal{V}$ is called the *derived bundle* of $\mathcal{V}$ and denoted by $\mathcal{V}'$.

For a Pfaffian system I with dual distribution $\mathcal{V}$ we have $I^{(1)} = (\mathcal{V}')^{\perp}$. By taking repeated derived bundles we arrive at the completion $\mathcal{V}^{\mathrm{compl}}$ of the bundle. This completion is integrable.

Definition 1.2.17. Given a distribution $\mathcal{V}$ we define the *Cauchy characteristic system* $C(\mathcal{V})$ as

$$C(\mathcal{V})_x = \{ X_x \in \mathcal{V}_x \mid X, Y \subset \mathcal{V},\ [X, Y] \subset \mathcal{V} \}.$$

A vector field X is a *Cauchy characteristic* vector field for $\mathcal{V}$ if $X \subset C(\mathcal{V})$. For smooth constant-rank distributions the Cauchy characteristic vector fields are precisely the vector fields X for which $[X, Y] \subset \mathcal{V}$ for all $Y \subset \mathcal{V}$. $\ominus$

Definition 1.2.18. We say a distribution $\mathcal{V}$ on M is in *involution* if for all vector fields $X, Y \subset \mathcal{V}$ we have $[X, Y] \subset \mathcal{V}$. A distribution $\mathcal{V}$ is in involution if and only if $C(\mathcal{V}) = \mathcal{V}$. A distribution is called *integrable* if locally there are coordinates $x^1, \dots, x^m$ such that the distribution is given by $\mathcal{V} = \mathrm{span}(\partial_{x^1}, \dots, \partial_{x^k})$. ⊖

Theorem 1.2.19 (Frobenius theorem). *Let $\mathcal{V}$ be a smooth constant-rank distribution of M. Then $\mathcal{V}$ is integrable if and only if $\mathcal{V}$ is in involution.*

If a distribution is integrable, then the maximal integral manifolds have dimension equal to the rank of the distribution. Locally these integral manifolds define a foliation of the manifold M. The individual integral manifolds are called the *leaves* of the distribution.

An *invariant* for a distribution $\mathcal{V}$ is a function I on M such that $X(I) = 0$ for all $X \subset \mathcal{V}$. This is equivalent to $\mathcal{V} \subset \ker(\mathrm{d}I)$. Classically, the invariants of a distribution are called *first integrals*. We say that m invariants $I^1, \dots, I^m$ are *functionally independent* at a point x if the rank of the Pfaffian system $\mathrm{span}(\mathrm{d}I^1, \dots, \mathrm{d}I^m)$ is equal to m at x. An integrable rank k distribution on an n-dimensional manifold has locally precisely $n - k$ functionally independent invariants. If $I^1, \dots, I^n$ are invariants of a distribution, then we will write $\{I^1, \dots, I^2\}_{\mathrm{func}}$ for the set of all functions that are functionally dependent with $I^1, \dots, I^n$.

Example 1.2.20. Consider the overdetermined first order system of partial differential equations for the function z of the variables x and p given by

$$z_x = -\alpha z, \quad z_p = -\beta z. \tag{1.5}$$

Here α and β are arbitrary functions of x and p. The first order jet bundle $\mathrm{J}^1(\mathbb{R}^2, \mathbb{R})$ has coordinates x, p, z, z_x, z_p and contact form

$$\theta = \mathrm{d}z - z_x \mathrm{d}x - z_p \mathrm{d}p.$$

Let M be the submanifold of the first order jet bundle defined by the two equations (1.5) and use x, z and p as coordinates on M. The solutions of the system (1.5) are locally in one-to-one correspondence with the 2-dimensional integral manifolds of the exterior differential on M generated by the single 1-form

$$\theta = \mathrm{d}z + \alpha z \mathrm{d}x + \beta z \mathrm{d}p$$

with independence condition $\Omega = \mathrm{d}x \wedge \mathrm{d}y$. The integral manifolds are precisely the integral manifolds of the distribution $\mathcal{V}$ dual to θ. We have

$$\begin{aligned} \mathrm{d}\theta &= \mathrm{d}(\alpha z) \wedge \mathrm{d}x + \mathrm{d}(\beta z) \wedge \mathrm{d}p \\ &= (\alpha_p z \mathrm{d}p + \alpha \mathrm{d}z) \wedge \mathrm{d}x + (\beta_x z \mathrm{d}x + \beta \mathrm{d}z) \wedge \mathrm{d}p \\ &\equiv (\alpha_p - \beta_x) z \mathrm{d}x \wedge \mathrm{d}p \mod \theta. \end{aligned}$$

The distribution is integrable at points where the compatibility condition $(\alpha_p - \beta_x)z = 0$ is satisfied. Since the system is linear, $z(x, p) = 0$ is always a solution. Near points where $(\alpha_p - \beta_x) \neq 0$ there are no other solutions to the system. At each point (x_0, p_0) where $\alpha_p - \beta_x = 0$ on a small neighborhood, the distribution is integrable. It follows from the Frobenius theorem that there is a unique integral manifold of the system through the point (x_0, p_0, z_0). ⊘

1.2.5 Lie brackets modulo the subbundle

Let $\mathcal{V}$ be a distribution on a smooth manifold M. Then the Lie brackets define a smooth map from $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ and by restriction a smooth map $\Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) \to \Gamma(TM)$. The value $[X, Y]_m$ at a point m depends not only on X_m and Y_m, but also on the first order derivatives of X and Y at m. For this reason the Lie brackets do not define a tensor, but are a first order differential operator. However, the value of $[X, Y]_m$ modulo $\mathcal{V}_m$ does not depend on the first order derivatives.

Lemma 1.2.21. *The Lie brackets of vector fields on M restrict to a tensor*

$$[\cdot, \cdot]_{/\mathcal{V}} : \mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}. \tag{1.6}$$

We call this tensor the Lie brackets modulo the subbundle*, and we often denote them as* $[\cdot, \cdot]_{/\mathcal{V}}$.

Proof. Let X, Y be smooth vector fields in $\mathcal{V}$ and assume that $X_m = 0$. Suppose that $\omega \in \Omega^1(M)$ and $\omega(\mathcal{V}) = 0$. Then $\omega(X) = \omega(Y) = 0$ and therefore

$$\begin{aligned}\omega_m([X, Y]_m) &= -(\mathrm{d}\omega)_m(X_m, Y_m) + X(\omega(Y))_m - Y(\omega(X))_m \\ &= -(\mathrm{d}\omega)_m(0, Y_m) = 0.\end{aligned}$$

So $\omega_m([X, Y]_m)$ is zero for all 1-forms ω dual to $\mathcal{V}$. This implies that $[X, Y]_m \in \mathcal{V}_m$ and hence that $[X, Y]_m \mod \mathcal{V}_m$ does only depend on the value of X at m and not on the first order derivative of X. By symmetry it follows that $[X, Y]_m$ also does not depend on the derivatives of Y. □

Since the Lie brackets restricted to $\mathcal{V} \times_M \mathcal{V}$ take values in the derived bundle $\mathcal{V}'$, the Lie brackets modulo the subbundle even define a tensor $\mathcal{V} \times_M \mathcal{V} \to \mathcal{V}'/\mathcal{V}$.

The Lie brackets modulo the subbundle give a simple characterization of the integral elements of a distribution (see Section 1.2.4). A k-plane $E \subset T_x M$ is an integral element for $\mathcal{V}$ if and only if $E \subset \mathcal{V}_x$ and the Lie brackets modulo $\mathcal{V}$ vanish when restricted to $E \times E$.

1.2.6 Projections and lifting

Let $\phi : M \to B$ be a smooth map. If ϕ is a diffeomorphism we can define the push forward $\phi_* X$ of a vector field X at $y = \phi(x)$ as $(\phi_* X)_y = (T_x\phi)X_x$. Locally we can define the push forward of a vector field under an immersion in the same way. If ϕ is a smooth map, then in general there is no push forward of a vector field X. The reason is that for two points x^1, x^2 with $\phi(x^1) = \phi(x^2) = y$ the vectors

$$(T_{x^1}\phi)(X) \quad \text{and} \quad (T_{x^2}\phi)(X)$$

might not be equal. If for all points x^1, x^2 with $\phi(x^1) = \phi(x^2)$ these vectors are equal, we say that X *projects down* to B and we write $\phi_* X$ for the projected vector field. In a similar way, we can project distributions $\mathcal{V}$ on M to B if for all points x in the fiber $\phi^{-1}(y)$ the image $(T_x\phi)(\mathcal{V})$ is equal to a fixed linear subspace $\mathcal{W}_{\phi(x)}$ of $T_{\phi(x)}B$.

Example 1.2.22. Let $\phi : \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first component. On $\mathbb{R}^2$ take coordinates x, y and define the vector fields

$$X = x\partial_x, \quad Y = x\partial_x + y\partial_y, \quad Z = (1 + y^2)\partial_x.$$

The vector fields X and Y project to the base manifold, the vector field Z does not project. The bundle $\mathcal{Z} = \operatorname{span}(Z)$ does project to $\mathbb{R}$. ⊘

Lemma 1.2.23 (Lie brackets of projected vector fields). *Let $\pi : M \to B$ be a smooth map. Let V, W be two vector fields on M that project to vector fields $v = \pi_* V$ and $w = \pi_* W$ on B, respectively. Then the commutator $[V, W]$ projects down to B and $\pi_*[V, W] = [v, w]$.*

Let $M \to B$ be a submersion. Given a vector field v on the base manifold B it is always possible to find a vector field V on the bundle M such that V projects down to v. The freedom we have in the choice of the vector field is precisely a section of the vertical bundle $\mathrm{V}(M)$. If we have a connection on M, i.e., a distribution on M of rank equal to $\dim B$ that is transversal to the projection, then there is a unique lift of every vector field below.

1.2.7 The frame bundle and G-structures

Definition 1.2.24. Let G be a Lie group and M a smooth manifold. A principal fiber bundle over M with structure group G is a fiber bundle $\pi : P \to M$ together with a smooth right action of G on P such that:

- G acts freely on P.
- M is the quotient space of P by the G-action, i.e., $M = P/G$.
- P is locally trivial. This means that for every point $x \in M$ there is a neighborhood U of x such that $\pi^{-1}(U)$ is isomorphic with $U \times G$. More precisely: locally there is a diffeomorphism $\tau : \pi^{-1}(U) \to U \times G$ such that $\pi|_{\pi^{-1}(U)} = \pi_1 \circ \tau$ where $\pi_1 : U \times G \to U$ is projection on the first component and τ intertwines the action of $\tilde{g} \in G$ on $\pi^{-1}(U)$ with the action $(x, h) \mapsto (x, h\tilde{g})$ on $U \times G$. ⊖

Furthermore, one has the following theorem [28, Theorem 1.11.4]: if G acts freely and properly on P, then there is a unique structure of a principal bundle $P \to M$ such that the action of G on P is the one of the principal bundle.

Definition 1.2.25. Let M be a smooth m-dimensional manifold. A *frame* on M, or more precisely a frame field, is an ordered set of vector fields $X_1, \ldots, X_m$ such that at every point $x \in M$ the vector fields form a basis of $T_x M$. In a similar way we define a *coframe* to be an ordered set of m linearly independent 1-forms. A *local frame* or *local coframe* on M is a frame or coframe defined on an open subset of M. ⊖

The map $X \mapsto \big((x, (c^1, \ldots, c^n)) \mapsto \sum_{j=1}^n c^j X_j(x)\big)$ is a bijection from the set of all frames on M to the set of all (inverses of) trivializations of TM.

Remark 1.2.26. The existence of a global coframe $\theta^1, \ldots, \theta^m$ implies the existence of a global non-vanishing m-form, i.e., $\theta^1 \wedge \ldots \wedge \theta^m$. This implies that the manifold is orientable and has trivial tangent bundle. Hence there are global obstructions against the existence of a global coframe on a manifold. ⊘

Since a coframe forms a basis for the differential 1-forms at each point, we can express the exterior derivative of the coframe differentials in terms of the coframe itself. We can write

$$\mathrm{d}\theta^i = 1/2 \sum_{j,k} T^i_{jk} \theta^j \wedge \theta^k = \sum_{j<k} T^i_{jk} \theta^j \wedge \theta^k \tag{1.7}$$

for unique anti-symmetric functions T^i_{jk}. The functions T^i_{jk} are called the *structure functions* of the coframe.

Definition 1.2.27. Let $E \to M$ be a smooth rank n vector bundle over M and V a fixed vector space of dimension n. The *frame bundle* $\mathrm{F}E$ of E is defined as the bundle over M for which the fiber $\mathrm{F}_x E$ over x is equal to the set of all linear isomorphisms $E_x \to V$. On $\mathrm{F}M$ we have a right $\mathrm{GL}(V)$ action defined by

$$G \times \mathrm{F}E \to \mathrm{F}E : (g, u) \mapsto g \cdot u = g^{-1} u.$$

The action is proper and free and exhibits $\mathrm{F}E$ as a principal $\mathrm{GL}(V)$-bundle. ⊖

The definition above is the definition of a V-valued frame bundle. When we refer to a frame bundle without explicitly mentioning V, we will assume $V = \mathbb{R}^n$. A point $b \in \mathrm{F}E$ will often be denoted by a pair (x, u) with $x \in M$ and $u \in \mathrm{Lin}(E_x, V)$. We will mainly use the frame bundle of the tangent space. For a manifold M we will use the notation $\mathrm{F}M$ to indicate the frame bundle $\mathrm{F}(TM)$. A section of $\mathrm{F}M$ defines both a framing and a coframing on M.

On the frame bundle there is a natural V-valued differential form called the *tautological 1-form* or *soldering form*. For a frame bundle $\pi : \mathrm{F}E \to M$ it is defined at a point $b = (x, u)$ by

$$\omega_b = u \circ T_b \pi. \tag{1.8}$$

If we choose a basis for V, then the components ω^j form a basis for the semi-basic forms on $\mathrm{F}M$.

Remark 1.2.28. Some authors [26] define the frame bundle of the tangent space analogously as the set of linear isomorphisms $f : \mathbb{R}^n \to T_x M$ (the right action is defined by $g \cdot f = f \circ g$). Sometimes the frame bundle is also defined as a basis for the tangent space [47, Section I.5] or as a set of equivalence classes for coordinate charts [63]. These definitions are all equivalent to our definitions. The only important choice one has to make is whether to use the right or the left action on the bundle. ⊘

Definition 1.2.29. Let G be a closed Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A *$G$-structure* on a smooth manifold M is a reduction of the frame bundle $\mathrm{F}M$ to a principal G-bundle $F \subset \mathrm{F}M$. Alternatively, a G-structure on M is defined by a section of the quotient bundle $\mathrm{F}M/G$. ⊖

For every diffeomorphism $M \to \tilde{M}$ there is a natural diffeomorphism $\Phi : \mathrm{F}M \to \mathrm{F}\tilde{M}$. For an element in $b : T_x M \to V \in \mathrm{F}M$ we define $\Phi(b)$ as the composition of the inverse of the tangent map $T_x M \to T_{\phi(x)}\tilde{M}$ with b. This gives the map $\tilde{b} = \Phi(b) : T_{\phi(x)}\tilde{M} \to V$ in $\mathrm{F}\tilde{M}$. This diffeomorphism is called the *lift* of ϕ or the *induced identification* of $\mathrm{F}M$ with $\mathrm{F}\tilde{M}$.

Definition 1.2.30. Let $B \to M$ and $\tilde{B} \to \tilde{M}$ be two G-structures. The two structures B and $\tilde{B}$ are *equivalent* if there is a diffeomorphism $\phi : M \to \tilde{M}$ such that the lifted map $\Phi : \mathrm{F}M \to \mathrm{F}\tilde{M}$ maps B to $\tilde{B}$.

If the G-structures B and $\tilde{B}$ are defined by sections $s : M \to \mathrm{F}(M)/G$ and $\tilde{s} : \tilde{M} \to \mathrm{F}(\tilde{M})/G$, respectively, then the G-structures are equivalent if there is a diffeomorphism ϕ such that $\Phi^*\tilde{s} = s$.

Proposition 1.2.31. *The two G-structures $\pi : B \to M$ and $\tilde{\pi} : \tilde{B} \to \tilde{M}$ are equivalent if and only if there exists a map $\Phi : B \to \tilde{B}$ such that $\Phi^*(\tilde{\omega}) = \omega$.*

Proof. If $\phi : M \to \tilde{M}$ is an equivalence of the G-structures, then the lift Φ satisfies the condition. Conversely, let Ψ be a map that preserves the soldering forms. The kernel of the soldering form is equal to the tangent space of the fibers $B \to M$. Hence any map Ψ that preserves the soldering forms must induce a map $\phi : M \to \tilde{M}$ such that the following diagram commutes.

$$\begin{tikzcd} B \arrow[r, "\Psi"] \arrow[d, "\pi"] & \tilde{B} \arrow[d, "\tilde{\pi}"] \\ M \arrow[r, "\phi"] & \tilde{M} \end{tikzcd}$$

Let $b = (x, u)$ and $\tilde{b} = \Psi(b) = (\tilde{x}, \tilde{u})$. Then

$$\omega_b = (\Psi^*\tilde{\omega})_b = \tilde{\omega}_{\tilde{b}} \circ T_b\Psi = \tilde{u} \circ T_{\tilde{b}}\tilde{\pi} \circ T_b\Psi = \tilde{u} \circ T_x\phi \circ T_b\pi.$$

At the same time $\omega_b = u \circ T_b\pi$ and hence $u = \tilde{u} \circ (T_x\phi)$. This implies $\tilde{u} = u \circ (T_x\phi)^{-1} = \Phi(u)$. □

Let $B \to M$ be a G-structure. For every point $b \in B$ we define the map $\mu_b : G \to B : g \mapsto g \cdot b$. The left-invariant Maurer-Cartan form on G is denoted by α_L.

Definition 1.2.32. A *connection form* for the G-structure B is a $\mathfrak{g}$-valued differential form γ on B such that for all $b \in B$, we have $\mu_b^*(\gamma) = \alpha_L$. ⊖

A *connection H* on the bundle $\pi : B \to M$ (here we mean bundle as a fiber bundle, without the additional G-structure) is a choice of complement H_b to the fibers of the bundle. At every point $b = (x, u)$ of the bundle we have $H_b \oplus T_u(B_x) = T_u B$. Suppose γ is a connection form on B. For every point $b \in B$ we can define $H_b = \ker \gamma$. This defines a connection on the bundle $B \to M$. Conversely, for any connection H we can define a unique connection form by requiring that for all points b in B we have $\ker \gamma_b = H_b$ and $\mu_b^*\gamma = \alpha_L$.

A connection form can have the additional property that the connection form is G-equivariant in the sense that $R_g^*(\gamma) = \mathrm{Ad}_{g^{-1}}(\gamma)$. The G-equivariance of γ is equivalent to the G-invariance of the corresponding connection H. See Duistermaat [26, Section 8] for more details. McKay [53] uses the terminology pseudoconnection for a connection and connection for a G-equivariant connection.

Proposition 1.2.33. *Let $B \to M$ be a G-structure with soldering form ω. Then the structure equations for ω can be written as*

$$\mathrm{d}\omega = -\gamma \wedge \omega + T(\omega \wedge \omega), \tag{1.9}$$

with γ a connection form for the G-structure. A $\mathfrak{g}$-valued form γ is a connection form if and only if the structure equation (1.9) *holds for a certain choice of torsion $T : \Lambda^2 V \otimes B \to V$.*

Example 1.2.34 (Connections). Let $M = \mathbb{R}^2$ with coordinates x, y. Let $G = \mathrm{SO}(2)$ and parameterize the elements of the group as

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

The left-invariant Maurer-Cartan form is given by

$$\alpha_L = \begin{pmatrix} 0 & -\mathrm{d}\phi \\ \mathrm{d}\phi & 0 \end{pmatrix}.$$

Let B be the G-structure defined by G and the coframe $(\mathrm{d}x, \mathrm{d}y)^T$. Then $B = \mathbb{R}^2 \times (\mathbb{R}/2\pi\mathbb{Z})$ and a point (x, y, ϕ) corresponds to the coframe $g^{-1}(\mathrm{d}x, \mathrm{d}y)^T$ at $(x, y) \in \mathbb{R}^2$. The soldering form is given by $\omega = g^{-1}(\mathrm{d}x, \mathrm{d}y)^T$ and the structure equations are $\mathrm{d}\omega = -\gamma \wedge \omega$. The form $\gamma = \alpha_L$ is the unique torsion-free connection form. The connection is G-equivariant.

Next consider the group G of diagonal matrices

diag(a, b). The left-invariant Maurer-Cartan form is given by

$$\alpha_L = \begin{pmatrix} a^{-1}\mathrm{d}a & 0 \\ 0 & b^{-1}\mathrm{d}b \end{pmatrix}.$$

The group defines a G-structure with soldering form given by $\omega = g^{-1}(\mathrm{d}x, \mathrm{d}y)^T$. The structure equations are $\mathrm{d}\omega = -\gamma \wedge \omega$. Here γ can be chosen as

$$\begin{pmatrix} a^{-1}\mathrm{d}a & 0 \\ 0 & b^{-1}\mathrm{d}b \end{pmatrix} + \begin{pmatrix} h_1\mathrm{d}x & 0 \\ 0 & h_2\mathrm{d}y \end{pmatrix}.$$

Here h_1 and h_2 are arbitrary functions. The connection form γ is G-equivariant if and only if the functions h_1 and h_2 depend only on x and y (and not on the group parameters a, b). $\oslash$

1.2.8 The Cartan-Kähler theorem

The Cartan-Kähler theorem is a very general existence theorem for solutions of analytic exterior differential systems. The theorem is explained and proved rigorously in Bryant et al. [13]. Some easier texts with examples are Olver [59] and Ivey and Landsberg [43]. There are basically two versions of the Cartan-Kähler theorem: a "non-linear" one for arbitrary exterior differential system and a "linear" one for linear Pfaffian systems. The linear version has the advantage that the difficult concepts of (regular) integral elements and integral flags can be replaced by a more algorithmic analysis of the structure equations of the system. The main disadvantage is of course that the linear version can only be used to analyze the linear Pfaffian systems.

In this Ph.D. thesis we will need only the linear version (except in one example), so therefore we present the linear version here. Another reason is that whenever we have a general exterior differential system the first prolongation of this system is a linear Pfaffian system. The presentation below uses the notation from Ivey and Landsberg [43]. Its purpose is to establish notation and to remind the reader of the different concepts involved.

Linear Pfaffian systems. Recall that a Pfaffian system I with independence condition Ω can equivalently be defined as two bundles (I, J) with rank $J/I = n$. If Ω is defined by $\Omega = \omega^1 \wedge \ldots \wedge \omega^n$, then the corresponding bundle J is defined by $J = \mathrm{span}(I, \omega^1, \ldots, \omega^n)$.

Definition 1.2.35 (Linear Pfaffian system). Let M be a smooth manifold and (I, J) a Pfaffian system with independence condition. The system (I, J) is called a *linear Pfaffian system* if $\mathrm{d}I \equiv 0 \mod J$. ⊖

The independence condition defines a natural affine structure in the space $\mathrm{Gr}_n(TM, \Omega)$ of n-planes E that satisfy $\Omega_E \neq 0$. The integral elements of a linear Pfaffian system with independence condition define affine linear subspaces of $\mathrm{Gr}_n(TM, \Omega)$, hence the name linear Pfaffian system. For a more detailed discussion see Bryant et al. [13, Chapter IV, §2].

Example 1.2.36 (Linear Pfaffian systems).

- Let M be a system of partial differential equations given as a submanifold of the jet bundle $\mathrm{J}^k(\mathbb{R}^n, \mathbb{R}^s)$. The pull back of the contact ideal on $\mathrm{J}^k(\mathbb{R}^n, \mathbb{R}^s)$ to M defines a linear Pfaffian system.

- Every prolongation of an exterior differential system with an independence condition is a linear Pfaffian system. ⊘

From here on we will choose a basis θ^a, $1 \leq a \leq s$ for the Pfaffian system I and forms ω^i, $1 \leq i \leq n$ that represent a basis for the bundle J/I. We will write down structure equations in terms of these bases and define concepts such as Cartan characters and prolongations in terms of these. A more geometric approach is also possible, but this would complicate the theory we need in this dissertation.

Definition 1.2.37 (Structure equations). Let (I, J) be a Pfaffian system with basis θ^a for I and basis ω^i for J/I. Choose 1-forms π^α such that $\theta^a, \omega^i, \pi^\alpha$ forms a basis of differential forms. The exterior derivatives of the forms θ^a are given by

$$\mathrm{d}\theta^a \equiv \pi_i^a \wedge \omega^i + T_{ij}^a \omega^i \wedge \omega^j + N_{\alpha\beta}^a \pi^\alpha \wedge \pi^\beta \mod I. \tag{1.10}$$

for unique functions $T_{ij}^a = -T_{ji}^a$, $N_{\alpha\beta}^a = -N_{\beta\alpha}^a$ and 1-forms $\pi_i^a = A_{\alpha i}^a \pi^\alpha$. The equations (1.10) are called the *structure equations* of the linear Pfaffian system. ⊖

A Pfaffian system is linear if and only if $N_{\alpha,\beta}^a = 0$. The terms $T_{ij}^a \omega^i \wedge \omega^j$ are sometimes written as $T^a(\omega \wedge \omega)$ and are called the *torsion* of the system. The torsion terms T_{ij}^a are not unique since we can always redefine the forms π^α. For this reason the torsion is called *apparent torsion*.

Example 1.2.38 (Absorption of torsion). Let $M = \mathbb{R}^4$ with coordinates x, y, z and p. Consider the Pfaffian system I generated by $\theta = \mathrm{d}z - p\mathrm{d}x - z\mathrm{d}y$ with independence condition $\Omega = \omega^1 \wedge \omega^2$, with $\omega^1 = \mathrm{d}x$, $\omega^2 = \mathrm{d}y$. Let $J = \mathrm{span}(I, \omega^1, \omega^2)$ and $\pi^1 = -\mathrm{d}p$. Then

$$\begin{aligned}\mathrm{d}\theta = -\mathrm{d}p \wedge \omega^1 - \mathrm{d}z \wedge \omega^2 &\equiv \pi^1 \wedge \omega^1 - p\omega^1 \wedge \omega^2 \mod I \\ &\equiv 0 \mod J.\end{aligned}$$

Hence (I, J) is a linear Pfaffian system. We can absorb the apparent torsion by redefining $\pi^1 = -\mathrm{d}p + p\omega^2$. Then

$$\mathrm{d}\theta \equiv \pi^1 \wedge \omega^1 \mod I. \tag{1.11}$$

⊘

Tableaux.

Definition 1.2.39. Let V and W be vector spaces. A *tableau* A is a linear subspace of $\mathrm{Hom}(V, W) = W \otimes V^*$.

Let $A \subset W \otimes V^*$ be a tableau and assume $\dim V = n$. A *flag* in V is a sequence of linear subspaces $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = V$ with $\dim V_i = i$. Any choice of basis v_j for V defines a flag by $V_j = \langle v_1, \ldots, v_j \rangle$. We define $A_k = \{\alpha \in A \mid \alpha(V_k) = 0\}$. We define $s_1 = \dim A - \dim A_1$ and then by induction $s_1 + \ldots + s_k = \dim A - \dim A_k$. The sum $s_1 + s_2 + \ldots + s_n$ is equal to the dimension of the tableau A. The numbers $s_1, \ldots, s_n$ are called the *characters* of the tableau with respect to the flag chosen.

For a generic flag the values of the $\dim A_k$ are minimal and hence the values of $s_1 + \ldots + s_k$ are maximal. The values of the characters s_k for a generic flag are called the *Cartan characters* of the tableau A. The Cartan characters satisfy $s_1 \geq s_2 \geq \ldots \geq s_n$ and are invariant under the action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on $W \otimes V^*$.

Remark 1.2.40. With respect to bases for V and W we can write the tableau A as a linear subspace of the space of $(s \times n)$-matrices. A tableau of dimension a can be represented by a $(s \times n)$-matrix with entries given by 1-forms on $\mathbb{R}^a$.

For a generic choice of basis for V the first Cartan character s_1 is equal to the number of independent entries in the first column of the matrix, $s_1 + s_2$ is equal to the number of independent entries in the first two columns of the matrix, etc. In examples this gives an easy method to determine the Cartan characters by looking at the matrix representation of the tableau. $\oslash$

Given a tensor product $V^*\otimes V^*$ there exists a canonical splitting $V^*\otimes V^* = S^2V^*\oplus\Lambda^2V^*$ into symmetric and anti-symmetric parts. We use this splitting to define for any tableau $A \subset W \otimes V^*$ the maps $\sigma : A \otimes V \to W \otimes S^2 V$ and $\delta : A \otimes V \to W \otimes \Lambda^2 V$ by

$$A \otimes V \to \left(W \otimes S^2 V^*\right) \oplus \left(W \otimes \Lambda^2 V^*\right) : x \mapsto \sigma(x) \oplus \delta(x).$$

The first prolongation $A^{(1)}$ of a tableau A is defined as the kernel of the map δ. We can write this symbolically as $A^{(1)} = A \otimes V^* \cap W \otimes S^2 V^*$. We can use a similar splitting of higher order tensor products $\otimes^l V^*$ to define the higher order prolongations of a tableau.

Definition 1.2.41 (Prolongation of a tableau). Let $A \subset W \otimes V^*$ be a tableau. We define the l-th *prolongation* of A by

$$A^{(l)} = \left(A \otimes (\otimes^l V^*)\right) \cap \left(W \otimes S^{(l+1)} V^*\right). \tag{1.12}$$

$\ominus$

Lemma 1.2.42. *Let A be a tableau with Cartan characters $s_1, s_2, \ldots, s_n$. Then*

$$\dim A^{(1)} \leq s_1 + 2s_2 + \ldots + ns_n. \tag{1.13}$$

Definition 1.2.43. A tableau A is *involutive* if $\dim A^{(1)} = s_1 + 2s_2 + \ldots + ns_n$. $\ominus$

Example 1.2.44 (Tableau). Let $V = \mathbb{R}^2$, $W = \mathbb{R}^2$ and define the tableau $A \subset W \otimes V^*$ by the set of matrices

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for $a, b \in \mathbb{R}$. The Cartan characters are $s_1 = 2$ and $s_2 = 0$. $\oslash$

Cartan-Kähler theorem. Let (I, J) be a linear Pfaffian system on M with basis θ^i, $1 \leq a \leq s$ for I, basis θ^a, ω^i, $1 \leq i \leq n$ for J and basis $\theta^a, \omega^i, \pi^\epsilon$ $(1 \leq \epsilon \leq \alpha)$ for the 1-forms on M. We have the structure equations

$$\mathrm{d}\theta^a = A^a_{\epsilon i}\pi^\epsilon \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j. \tag{1.14}$$

At each point x the coefficients $A^a_{\epsilon i}(x)$ define a tableau $A_x \subset W \otimes V^*$ with $W = I^*_x$, $V = (J/I)^*_x$. The tableau is spanned by elements of the form $A^a_{\epsilon j}\partial_{\theta^a} \otimes \omega^j$.

The torsion T^a_{ij} defines a section T of the bundle $W \otimes \Lambda^2 V^* = I^* \otimes \Lambda^2(J/I)$. By redefining a form π^ϵ we can change the apparent torsion. If we redefine $\pi^\epsilon \mapsto \pi^\epsilon + e^\epsilon_i \omega^i$, then

$$A^a_{\epsilon j} \pi^\epsilon \wedge \omega^j \mapsto A^a_{\epsilon j} \pi^\epsilon \wedge \omega^j + A^a_{\epsilon j} e^\epsilon_i \omega^i \wedge \omega^j .$$

Hence the apparent torsion is changed by a term $A^a_{\epsilon j} e^\epsilon_i \omega^i \wedge \omega^j$. The freedom we have to change the apparent torsion by modifying the forms π^ϵ, is precisely equal to $\delta(A \otimes V^*)$. Using this freedom to eliminate the apparent torsion is called *absorption of torsion*. See Example 1.2.38 for a small example. The quotient of the torsion bundle $I^* \otimes \Lambda^2(J/I)$ by the image of δ is $H^{(0,2)}(A) = I^* \otimes \Lambda^2(J/I)/\operatorname{im}\delta$ and is called the *intrinsic torsion* [1]. The torsion T induces a section $[T]$ of the intrinsic torsion bundle. At points where $[T] \neq 0$ there exist no integral elements. Hence the vanishing of the intrinsic torsion is a necessary condition for the existence of integral manifolds.

If A_x is the tableau associated to a linear Pfaffian system, then the first prolongation $A^{(1)}_x$ is isomorphic to the space $V_n(I)_x$ of integral elements at the point x. We say the linear Pfaffian system is in *involution* at x if the corresponding tableau A_x is in involution. If the system is in involution, then the Cartan characters of the tableau A_x are locally constant and we define the Cartan characters of the linear Pfaffian system as the characters of the tableau A_x. Theorem 1.2.45 below shows that the involutivity of the tableau (which is an algebraic property) together with the vanishing of the torsion implies that the corresponding system of partial differential equation is in involution (there are no hidden integrability conditions) and there exist n-dimensional integral manifolds. The use of Definition 1.2.43 to check whether a linear Pfaffian system is in involution or not is called *Cartan's test*.

Theorem 1.2.45 (Cartan-Kähler theorem for linear Pfaffian systems). *Let (I, J) be an analytic linear Pfaffian system on M with* rank $J/I = n$. *Let $x \in M$ and U a neighborhood of x such that:*

- *The intrinsic torsion vanishes, i.e.,* $[T] = 0$.
- *The system is in involution.*

Then there exist integral manifolds of dimension n through the point x that depend on s_l functions of l variables.

Example 1.2.46 (continuation of Example 1.2.38). Consider the linear Pfaffian system defined in Example 1.2.38 with the structure equations (1.11). The tableau for this system is given by the (1×2)-matrices

$$\begin{pmatrix} \pi^1 & 0 \end{pmatrix} .$$

The Cartan characters are $s_1 = 1$ and $s_2 = 0$. The first prolongation has dimension one and the tableau is in involution. By the Cartan-Kähler theorem the integral manifolds of the

[1] The notation $H^{(0,2)}(A)$ is inspired by the fact that the intrinsic torsion is part of the Spencer cohomology groups of the tableau. See Ivey and Landsberg [43, pp. 180–181].

system can be parameterized by one function of one variable. Indeed, the integral manifolds can be given in parametric form as

$$z(x, y) = c \exp(y) + \phi(x), \quad p(x, y) = \phi'(x),$$

with c a constant and ϕ an arbitrary function. ⊘

1.2.9 Clean intersections

We will formulate the regularity conditions for the equivalence of coframes in terms of clean intersections. The terminology was introduced by Raoul Bott [11, pp. 194–199] and is explained in Duistermaat and Guillemin [27, p. 63].

Let X, Y, Z be smooth manifolds and $f : X \to Z$, $g : Y \to Z$ smooth maps. We can then form the fibered product

$$F = \{ (x, y) \in X \times Y \mid f(x) = g(y) \}.$$

The diagram below is useful to keep in mind.

$$\begin{tikzcd} F \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

We define the map $\tau : X \times Y \to Z \times Z : (x, y) \mapsto (f(x), g(y))$ and the diagonal $\Delta = \{ (z, z) \in Z \times Z \}$. We say that the maps f and g have a *clean intersection* at $p = (x, y)$ if F is a smooth submanifold of $X \times Y$ at p and $T_p F$ equals $\{ (V, W) \in T_p(X \times Y) \mid T_x f(V) = T_y g(W) \}$. Another formulation of this last condition is that the tangent space $T_p F$ is equal to $(T_p \tau)^{-1}(T_{(f(x),g(y))}\Delta)$. Instead of saying that f and g intersect cleanly we can also say that τ intersects the diagonal Δ cleanly at p. We say that f and g have a clean intersection if f and g intersect cleanly at all points $p \in F$.

A clean intersection is a generalization of a transversal intersection. If two maps $f_0 : X \to Z_0$ and $g_0 : X \to Z_0$ have transversal intersection and Z_0 is a submanifold of Z embedded as $\iota : Z_0 \to Z$, then the maps $f : X \to Z = \iota \circ f_0$ and $g : X \to Z : \iota \circ g$ have a clean intersection. Not all clean intersection have the nice structure of a transversal intersection embedded in a higher dimensional manifold, see Example 1.2.48.

Example 1.2.47 (Clean intersection). Let $X = \mathbb{R}$, $Y = \mathbb{R}$ and $Z = \mathbb{R}^3$. Let $f : X \to Z$ and $g : Y \to Z$ be two curves intersecting at a point z, i.e., $z = f(x) = g(y)$ for certain $x \in X$ and $y \in Y$. If $f'(x) \neq 0$, $g'(y) \neq 0$ and $f'(x) \neq g'(y)$, then the curves f and g have a clean intersection. The linear space at z spanned by $T_x f(T_x X)$ and $T_y g(T_y Y)$ has codimension one in $T_z Z$ and hence the intersection is not transversal. ⊘

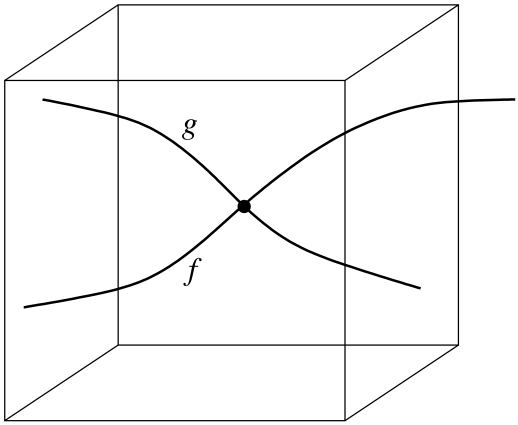

Figure 1.1: Clean intersection of two space curves (Example 1.2.47)

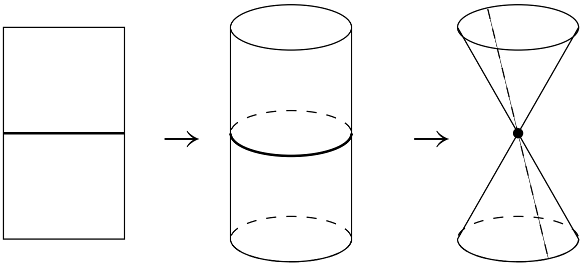

Figure 1.2: Clean intersection that is singular (Example 1.2.48)

Example 1.2.48 (Singular clean intersection). Let $X = \mathbb{R}^2$ be a plane. We map X to a cylinder in $\mathbb{R}^3$ using the map $(x_1, x_2) \mapsto (\cos(x_1), \sin(x_1), x_2)$. Then we map this cylinder in $\mathbb{R}^3$ to a cone in $\mathbb{R}^3$ using the map $(z_1, z_2, z_3) \mapsto (z_1 z_3, z_2 z_3, z_3)$. The composition gives the map

$$f : X \to \mathbb{R}^3 : (x_1, x_2) \mapsto (\cos(x_1)x_2, \sin(x_1)x_2, x_2),$$

that maps X to a cone and the line $x_2 = 0$ to the singular point $(0, 0, 0)$. Let $Y = \mathbb{R}^0$ and take the map $g : Y \to Z : 0 \mapsto (0, 0, 0)$. Then $F \subset X \times Y$ consists of the product of the line $x_2 = 0$ in X and the single point 0 in Y. The intersection of f and g is clean even though the image of X is not a submanifold at $z = (0, 0, 0)$. $\oslash$

1.2.10 Equivalence of coframes

In this section we will analyze the equivalence of coframes under general diffeomorphisms. The notation in this section and the definition of classifying space and structure map are taken from Olver [59, Chapter 8]. The main Theorem 1.2.54 on the equivalence of coframes is stated and proved under weaker conditions then the equivalent theorem in [59].

Definition 1.2.49. Let $\theta^1, \ldots, \theta^m$ be a coframe on M and $\tilde{\theta}^1, \ldots, \tilde{\theta}^m$ a coframe on $\tilde{M}$. The equivalence problem for coframes is to decide whether these exists a (local) diffeomorphism $\Phi : M \to \tilde{M}$ such that

$$\Phi^* \tilde{\theta}^i = \theta^i, \quad 1 \le i \le m. \tag{1.15}$$

$\ominus$

We will write $\mathrm{d}\theta^i = T^i_{jk} \theta^j \wedge \theta^k$ for the structure equations of θ and similar for $\tilde{\theta}$. Because the exterior derivative operator d commutes with the pullback Φ^*, the formula (1.15) implies that $\Phi^*(\mathrm{d}\tilde{\theta}^i) = \mathrm{d}\theta^i$. In terms of the structure functions for the two coframes this equation becomes

$$\tilde{T}^i_{jk}(\Phi(x)) \Phi^* \tilde{\theta}^j \wedge \Phi^* \tilde{\theta}^j = T^i_{jk}(x) \theta^j \wedge \theta^k.$$

This implies that $\Phi^* \tilde{T}^i_{jk} = T^i_{jk}$. From this last equation we see that the structure functions of a coframe are very important when determining the equivalence class of the coframe. In particular if the frames θ^i and $\tilde{\theta}^i$ are equivalent and T^i_{jk} is constant in a neighborhood of x then $\tilde{T}^i_{jk}$ is constant in a neighborhood of $\Phi(x)$. So if the coframe on M has constant structure functions and the coframe on $\tilde{M}$ has non-constant structure functions, then these coframes cannot be equivalent.

Example 1.2.50. Let $M = \mathbb{R}^2$ with coordinates (x, y) and define two coframes $\theta^1 = \mathrm{d}x$, $\theta^2 = \mathrm{d}y$ and $\tilde{\theta}^1 = \mathrm{d}x$, $\tilde{\theta}^2 = (1 + x^2)\mathrm{d}y$. Then we have $\mathrm{d}\theta^1 = \mathrm{d}\theta^2 = \mathrm{d}\tilde{\theta}^1 = 0$, $\mathrm{d}\tilde{\theta}^2 = x\mathrm{d}x \wedge \mathrm{d}y$. From this we can read off the structure functions

$$T^1_{12} = T^2_{12} = 0,$$
$$\tilde{T}^1_{12} = 0, \quad \tilde{T}^2_{12} = \frac{1}{2} x.$$

The other structure functions follow by the asymmetry properties. The first frame has constant structure functions. The second frame will have non-constant structure functions in *any* coordinate system and therefore both frames are non-equivalent. $\oslash$

By making a careful analysis of the structure functions of a coframe one can completely solve the equivalence problem for coframes. Below we will define the concepts necessary to formulate Theorem 1.2.54 and Theorem 1.2.58. These theorems are the main theorems we need in this dissertation.

Definition 1.2.51. We define the *coframe derivative* $\partial I / \partial \theta^j$ with respect to a coframe θ^j of a function I by

$$\mathrm{d}I = \frac{\partial I}{\partial \theta^j} \theta^j. \tag{1.16}$$

If X_j is the frame dual to θ^j, then $\partial I/\partial\theta^j = X_j(I) = \mathrm{d}I(X_j)$. Note that the coframe derivatives do not necessarily commute since the coframes are not always derived from coordinate coframes. ⊖

Definition 1.2.52. Let M be an m-dimensional manifold and θ^j, $1 \leq j \leq m$ a coframe on M. For a multi-index $\sigma = (i, j, k, l_1, \ldots, l_s)$ with $1 \leq i, j, k \leq m$, $0 \leq s$ we define the structure invariant T_σ as

$$T_\sigma = \frac{\partial^s T^i_{jk}}{\partial\theta^{l_s}\partial\theta^{l_{s-1}}\cdots\partial\theta^{l_1}}. \tag{1.17}$$

The integer s is called the order of σ. ⊖

The multi-indices σ with the properties described in the definition we call the *structure indices*. The T_σ with σ of order s form the most general structure invariants of order s corresponding to the coframe. The structure invariants are invariants for the coframe just as the structure functions. In order for two coframes to be equivalent we need the structure invariants to be the same. The converse is also true, with some regularity assumptions on the coframe: if all the structure invariants are the same, then the corresponding coframes are equivalent.

There are many relations between the structure functions. The knowledge of all structure equations T_σ for which $s \leq S$ and

$$j < k, \quad 1 \leq l_1 \leq \ldots \leq l_s \leq m, \tag{1.18}$$

is sufficient to determine all structure functions of order at most S. This follows from the fact that the T^i_{jk} are anti-symmetric in j, k and the commutation relations $[\partial_{\theta^j}, \partial_{\theta^k}] = -2T^i_{jk}\partial_{\theta^i}$. The collection of structure indices of order $s \leq S$ that satisfy (1.18) is denoted by I_S. The number of structure indices in I_S is $q_s(m) = \frac{1}{2}m^2(m-1)\binom{m+s}{m}$.

Definition 1.2.53. Let M be an m-dimensional manifold with coframe θ. The *s-th order classifying space* $\mathbb{K}^{(s)}$ of M is the $q_s(m)$-dimensional Euclidean space $\mathbb{R}^{q_s(m)}$. On the classifying space we introduces coordinates z_σ, where σ is a structure index in I_s. The *s-th order structure map* is defined as

$$T^{(s)} : M \to \mathbb{K}^{(s)} : x \mapsto T_\sigma(x), \sigma \in I_S.$$

Let ρ_s denote the rank of the s-th order structure map. ⊖

Let θ and $\tilde\theta$ be two coframes on M and $\tilde M$, respectively. If $T^{(s)}(x) = \tilde T^{(s)}(\tilde x)$, then we have by definition $T_\sigma = \tilde T_\sigma$ for all structure indices $\sigma \in I_s$. Using the commutation relations between the coframe derivatives we can then prove that $T_\sigma = \tilde T_\sigma$ for all structure indices σ or order at most s. For example consider the two structure indices $\sigma = (i, j, k, 1, 2)$ and $\tau = (i, j, k, 2, 1)$. Both indices have order 2 but $\sigma \in I_2$, while $\tau \notin I_2$. However

$$\begin{aligned} T_\tau = \frac{\partial^2 T^i_{jk}}{\partial\theta^1\theta^2} &= X^1 X^2 T^i_{jk} = X^2 X^1 T^i_{jk} + [X_1, X_2]T^i_{jk} \\ &= T_\sigma - 2T^l_{12}X_l T^i_{jk}. \end{aligned}$$

So the structure function with index τ has been expressed in structure functions with indices in I_2.

The structure map of a coframe almost completely describes the properties of the coframe. Together with some regularity conditions on the coframe we can prove the main theorem.

Theorem 1.2.54 (Equivalence of coframes). *Let θ and $\tilde{\theta}$ be coframes on n-dimensional manifolds M and $\tilde{M}$, respectively. Let $Z_s = \mathbb{K}^{(s)} = \mathbb{R}^{q_s(n)}$ and define*

$$\tau_s : M \times \tilde{M} \to Z_s \times Z_s : (x, \tilde{x}) \mapsto (T^{(s)}(x), \tilde{T}^{(s)}(\tilde{x})). \tag{1.19}$$

In the product $Z_s \times Z_s$ we define the diagonal $\Delta_s = \{ (z, z) \in Z_s \times Z_s \}$.

Assume that for a certain order s the two following conditions are satisfied:

- *The map τ_s has a clean intersection with Δ_s. In other words, the two structure maps $T^{(s)}$ and $\tilde{T}^{(s)}$ have a clean intersection. This implies that $F_s = \tau_s^{-1}(\Delta_s)$ is a smooth submanifold of $M \times \tilde{M}$.*
- *The manifolds F_s and F_{s+1} are equal and non-empty, so there is point $f = (x, \tilde{x}) \in F_s$.*

Then there exists a local equivalence ϕ between the two coframes θ and $\tilde{\theta}$ with $\phi(x) = \tilde{x}$.

Proof. Note that $T_{(x,\tilde{x})}(M \times \tilde{M})$ is canonically equivalent to $T_x M \times T_{\tilde{x}} \tilde{M}$. We start with the definition of a distribution $\mathcal{V}$ on $M \times \tilde{M}$. The distribution is spanned by all pairs of vectors (V, W) in $T_x M \times T_{\tilde{x}} \tilde{M}$ such that V and W have the same trivialization. In formula:

$$\mathcal{V}_{(x,\tilde{x})} = \{ (V, W) \in T_{(x,\tilde{x})}(M \times \tilde{M}) \mid \theta^j_x(V) = \tilde{\theta}^j_{\tilde{x}}(W),\ 1 \le j \le n \}.$$

Another description of $\mathcal{V}$ is that $\mathcal{V}$ is spanned by the vector fields $(X_j, \tilde{X}_j)$ in $T(M \times \tilde{M})$. Here X_j and $\tilde{X}_j$ are the dual frames to θ^j and $\tilde{\theta}^j$, respectively. From the definition it is clear that $\mathcal{V}$ has rank n.

We claim that at all points $(x, \tilde{x}) \in F_s$ the distribution $\mathcal{V}$ is contained in $T_{(x,\tilde{x})} F_s$. Let σ be a structure index of order s or smaller. The condition that $F_s = F_{s+1}$ together with the remarks on page 22 on changing the order of coframe derivatives imply that

$$\mathcal{L}_{X_j} T_\sigma(x) = T_{\sigma,j}(x) = \tilde{T}_{\sigma,j}(\tilde{x}) = \mathcal{L}_{\tilde{X}_j} \tilde{T}_\sigma(\tilde{x}), \tag{1.20}$$

for all $\sigma \in I_s$. But this implies that the vector field $(T_{(x,\tilde{x})}\tau)(X_j, \tilde{X}_j)$ is tangent to the diagonal Δ_s in Z_s. The condition that τ_s has clean intersection with Δ_s implies that $(X_j, \tilde{X}_j)$ is contained in the tangent space to F_s.

Since $\mathcal{V}$ is contained in $T F_s$ this defines a rank n distribution on F_s. The commutator of two vector fields in $\mathcal{V}$ is

$$[(X_j, \tilde{X}_j), (X_k, \tilde{X}_k)] = ([X_j, X_k], [\tilde{X}_j, \tilde{X}_k]) = (T^i_{jk} X_i, \tilde{T}^i_{jk} \tilde{X}_i). \tag{1.21}$$

On F_s the values of $T^{(s)}$ and $\tilde{T}^{(s)}$ are equal and therefore the values of T^i_{jk} and $\tilde{T}^i_{jk}$ are equal as well. Hence

$$[(X_j, \tilde{X}_j), (X_k, \tilde{X}_k)] = T^i_{jk}(X_i, \tilde{X}_i) \subset \mathcal{V}. \tag{1.22}$$

This proves that $\mathcal{V}$ is integrable. The definition of $\mathcal{V}$ also makes clear that $\mathcal{V}$ is transversal to the projections $M \times \tilde{M} \to M$ and $M \times \tilde{M} \to \tilde{M}$. The integral manifolds $\mathcal{V}$ can be found using the Frobenius theorem. These integral manifolds are precisely the graphs of (local) equivalences between θ and $\tilde{\theta}$. □

Definition 1.2.55. A coframe θ on M is *regular* of rank r if for some $s \geq 0$ the ranks of $T^{(s)}$ and $T^{(s+1)}$ are equal to r and constant on M. The smallest integer s for which this condition holds is called the *order* of the coframe. A coframe is called *fully regular* if for all orders $s \geq 0$ the structure map $T^{(s)}$ is regular. ⊖

If a coframe is regular, then the image of the s-th order structure map is a submanifold of $\mathbb{K}^{(s)}$ and we speak of the classifying manifold.

Example 1.2.56.

- The standard coframe $\theta^1 = \mathrm{d}x$, $\theta^2 = \mathrm{d}y$ on $\mathbb{R}^2$ is a fully regular coframe of rank 0 (and therefore of order 0).
- The coframe $\theta^1 = \exp(y)\mathrm{d}x$, $\theta^2 = \mathrm{d}y$ on $\mathbb{R}^2$ has structure equations
$$\mathrm{d}\theta^1 = -\theta^1 \wedge \theta^2, \quad \mathrm{d}\theta^2 = 0.$$
The coframe is fully regular, has rank 0 and has order 0.
- The coframe $\theta^1 = \mathrm{d}x + \exp(x)\mathrm{d}y$, $\theta^2 = \mathrm{d}y$ on $\mathbb{R}^2$ has structure equations
$$\mathrm{d}\theta^1 = \exp(x)\theta^1 \wedge \theta^2, \quad \mathrm{d}\theta^2 = 0.$$
The only non-zero structure function of order 0 is $J = T^1_{12} = \exp(x)$. To find the rank and order of the coframe we calculate the structure invariants of order 1.
$$\frac{\partial J}{\partial \theta^1} = \exp(x), \quad \frac{\partial J}{\partial \theta^2} = (\exp(x))^2.$$
These structure invariants are functionally dependent on J. The rank of $T^{(1)}$ is 1. The coframe is fully regular with order 0 and rank 1. ⊘

Definition 1.2.57. A local symmetry of a coframe θ on a manifold M is a local diffeomorphism $\phi : M \to M$ such that $\phi^*(\theta^j) = \theta^j$, $j = 1, \ldots, m$. ⊖

For sufficiently regular coframes the symmetry group is a finite-dimensional local Lie group.

Theorem 1.2.58 (Symmetry group of coframe). *If θ is a regular coframe of rank r on an m-dimensional manifold M, then the symmetry group of θ is a local Lie group of dimension $(m - r)$.*

Proof. This is Theorem 14.16 in Olver [59]. We give a proof based on Theorem 1.2.54 above. We adopt the same notation as in the theorem.

Let θ be a coframe on M and let $\tilde{\theta}$ be a copy of θ on $\tilde{M} = M$. The condition that the coframe is regular implies that for some order s the structure map $T^{(s)}$ is a regular map with rank r. This means the image of $T^{(s)}$ is a smooth r-dimensional submanifold U of Z_s.

Choose a point $(x, \tilde{x}) \in M \times \tilde{M}$ with $x = \tilde{x}$. Since both $T^{(s)}$ and $\tilde{T}^{(s)}$ have the same image U we can consider the map $\eta : M \times \tilde{M} \to U : (x, \tilde{x}) \mapsto T^{(s)}(x) - \tilde{T}^{(s)}(\tilde{x})$. The map η is a smooth submersion and hence locally the inverse image $F_s = \eta^{-1}(0)$ is a smooth codimension r submanifold of $M \times \tilde{M}$. The intersection of τ_s with the diagonal is a clean intersection. This follows from the fact that $T_x T^{(s)}(T_x M) = T_u U = T_{\tilde{x}} \tilde{T}^{(s)}(T_{\tilde{x}} \tilde{M})$.

From the definitions it follows that $F_{s+1} \subset F_s$. Since by assumption $T^{(s+1)}$ is a regular map of order r, also F_{s+1} as a submanifold of dimension $2n - r$ and hence $F_s = F_{s+1}$. Hence all the conditions in Theorem 1.2.54 are satisfied. The local equivalences form a group and correspond to the integral manifolds of the distribution $\mathcal{V}$. Since $\mathcal{V}$ is a rank n distribution on the $(2n - r)$-dimensional submanifold F_s, the space of integral manifolds is a manifold of dimension $n - r$. □

Example 1.2.59. Let θ be a regular coframe of rank zero on an open subset M of $\mathbb{R}^2$. Then either

$$\mathrm{d}\theta^1 = \mathrm{d}\theta^2 = 0$$

and the symmetry group is the (local) 2-dimensional translation group of $\mathbb{R}^2$, or

$$\mathrm{d}\theta^1 = \alpha\theta^1 \wedge \theta^2, \quad \mathrm{d}\theta^2 = \beta\theta^1 \wedge \theta^2$$

with $(\alpha, \beta) \neq (0, 0)$ and the symmetry group has Lie algebra isomorphic to $\mathfrak{aff}(1)$, the affine group.

For example take $\theta^1 = \mathrm{d}x$, $\theta^2 = \mathrm{d}y + y\mathrm{d}x$. Then $\mathrm{d}\theta^1 = 0$ and $\mathrm{d}\theta^2 = -\theta^1 \wedge \theta^2$. The infinitesimal symmetries are spanned by ∂_x and $\partial_x + \exp(-x)\partial_y$. ⊘

1.2.11 The method of equivalence

In the previous section we have analyzed the equivalence of coframes. Although we could (in principle) give a complete solution to the problem, there are many problems that cannot be formulated as an equivalence problem for coframes. Very often a geometric structure can be formulated in terms of a coframe and a structure group acting on this frame. If this is the case, then we can apply the *method of equivalence* to solve the equivalence problem for the structure. In this section we present an outline of the main steps in the method. The method was introduced by Cartan, see [19] for example. The method was given a proper formulation in Gardner [36] and Gardner [37].

Geometric formulation

Example 1.2.60 (Equivalence of Riemannian metrics). Let $\mathrm{d}s^2$ and $\mathrm{d}\tilde{s}^2$ be Riemannian metrics on the manifolds M and $\tilde{M}$ respectively. First we note that since $\mathrm{d}s^2$ and $\mathrm{d}\tilde{s}^2$ are positive definite, symmetric forms, we can introduce local coframes $\omega^1, \ldots, \omega^m$ and $\tilde{\omega}^1, \ldots, \tilde{\omega}^m$ on M and $\tilde{M}$ such that the metrics are given by

$$\mathrm{d}s^2 = \sum_i (\omega^i)^2, \quad \mathrm{d}\tilde{s}^2 = \sum_i (\tilde{\omega}^i)^2.$$

The local equivalence of two metrics is then equivalent to the existence of a (local) diffeomorphism $\Phi : M \to \tilde{M}$ such that

$$\Phi^*(\tilde{\omega}^i) = g^i_j(x)\omega^j$$

for an $(m \times m)$-matrix valued function g that takes values in the orthogonal group $\mathrm{O}(m)$. ⊘

Example 1.2.61 (Equivalence of differential equations). Consider a first order ordinary differential equation defined by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = F(x, z).$$

On the first order jet bundle of $\mathbb{R}^2$ we have coordinates x, y, p and the contact form $\theta = \mathrm{d}z - p\mathrm{d}x$. Let M be the hypersurface in the jet bundle defined by the equation $p = F(x, z)$. We define the two forms

$$\theta^1 = \mathrm{d}z - F\mathrm{d}x, \quad \theta^2 = \mathrm{d}x.$$

The form θ^1 is the pullback of the contact form θ on $\mathrm{J}^1(\mathbb{R}^2)$. Solutions of the differential equation are in one-to-one correspondence with integral surfaces of θ^1. Since the contact form θ^1 can be scaled by arbitrary functions, the structure of our differential equation is encoded in the coframe θ^1, θ^2 with structure group

$$G = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, c \neq 0 \right\}. \qquad ⊘$$

In the two examples above we could describe a geometric structure by specifying a particular coframe θ on a manifold M and a structure group G. The structure group is assumed to be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. The coframe and the structure group G together determine a G-structure on M. The set of all coframes obtained from θ by multiplication with an element in G forms a principal G-bundle in the frame bundle $\mathrm{F}M$.

An equivalence of two geometric structures defined by a coframe θ on M and a coframe $\tilde{\theta}$ on $\tilde{M}$ is an equivalence of the corresponding G-structures B and $\tilde{B}$. This is a diffeomorphism $\phi : M \to \tilde{M}$ such that

$$\phi^*(\tilde{\theta}) = g\theta, \tag{1.23}$$

where g is a function valued in the structure group G.

On B we have the pullback of the soldering form. This pullback is equal to $\omega = g^{-1}\theta$. Every equivalence $\phi : M \to \tilde{M}$ lifts to a map $\Phi : B \to \tilde{B}$. Then the condition

$$\phi^*(\tilde{\theta}) = g\theta \tag{1.24}$$

is equivalent to

$$\Phi^*(\tilde{\omega}) = \omega. \tag{1.25}$$

Action on the invariants

Let B be a G-structure with soldering form ω. The structure equations for ω can be written as

$$\mathrm{d}\omega = -\gamma \wedge \omega + T(\omega \wedge \omega).$$

These structure equations are very similar to the structure equations for linear Pfaffian systems. The torsion T depends on the structure group and we can normalize some of the torsion coefficients. If this is possible the structure group reduces to a smaller structure group and we can write down the reduced structure equations.

The structure group induces an action on the invariants. We can either calculate this action directly (the *parametric method*), or calculate the infinitesimal action using the structure equations. The calculation of the infinitesimal action using $\mathrm{d}^2\omega = 0$ is called the *intrinsic method* (see Olver [59, pp. 358–361], Brockett et al. [12, pp. 168–170] and Gardner [37]).

Technique of the graph

Let B be a G-structure with structure equations for the soldering form ω given by

$$\mathrm{d}\omega = -\gamma \wedge \omega + T(\omega \wedge \omega).$$

We assume the torsion T is constant. Then we can solve the equivalence problem using the Cartan-Kähler theorem. Let $\tilde{B} \to \tilde{M}$ be a copy of $B \to M$. Every symmetry of G-structures $\phi : M \to \tilde{M}$ lifts to a map $\Phi : B \to \tilde{B}$ that matches the soldering forms on B and $\tilde{B}$. The graph of such a symmetry is an integral manifold of $B \times \tilde{B}$ for the exterior differential system generated by $\Omega = \omega - \tilde{\omega}$. The graphs of equivalences all satisfy the independence condition that $\omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^n \neq 0$. The assumption of constant torsion implies that

$$\begin{aligned} \mathrm{d}\Omega &= -\gamma \wedge \omega - \tilde{\gamma} \wedge \tilde{\omega} \\ &\equiv -(\gamma - \tilde{\gamma}) \wedge \omega \mod \Omega. \end{aligned}$$

This is a linear Pfaffian system with zero torsion. If the system is in involution, then there exist integral manifolds by the Cartan-Kähler theorem. The involutivity condition only depends on the structure of the Lie group G. The involutivity can be checked by calculating the Cartan characters and the dimension of the first prolongation of the Lie algebra of G.

Prolongation

To apply the Cartan-Kähler theorem the system needs to be in involution. If the system is not in involution we can prolong the equivalence problem. For details on this prolongation (which should not be confused with the prolongation of the exterior differential system itself) we refer to Olver [59, Chapter 12].

Example 1.2.62 (Conformal geometry). Consider a Riemannian metric on $\mathbb{R}^2$ given by $\mathrm{d}s^2 = (\theta^1)^2 + (\theta^2)^2$. We want to analyze the possible equivalences of this metric under conformal transformations. The group $\mathrm{CO}(2, \mathbb{R})$ is the group of matrices of the form $g = \lambda S$ with $S \in \mathrm{SO}(2, \mathbb{R})$, $\lambda \neq 0$. If we use λ and ϕ as parameters (ϕ is the usual angle of rotation for $\mathrm{SO}(2, \mathbb{R})$), then the left-invariant and right-invariant Maurer-Cartan forms are equal and are given by

$$\alpha_L = \alpha_R = \begin{pmatrix} \mathrm{d}\lambda/\lambda & -\mathrm{d}\phi \\ \mathrm{d}\phi & \mathrm{d}\lambda/\lambda \end{pmatrix}.$$

The structure equations for the lifted coframe $\omega = g^{-1}\theta$ on $B = \mathbb{R}^2 \times \mathrm{CO}(2, \mathbb{R})$ become

$$\mathrm{d}\omega = -\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \wedge \omega,$$

for certain one-forms α, β. All torsion has been absorbed by a suitable choice of α, β, but there are still two group parameters. The Cartan characters are $s_1 = 2$ and $s_2 = 0$ and the dimension of the first Lie algebra prolongation is 2 (see Example 1.2.44). Therefore Cartan's test is satisfied and the system is in involution. We conclude that in the analytic setting all metrics on $\mathbb{R}^2$ are conformally equivalent. The symmetry group of a given conformal metric depends on two functions of one variable. $\oslash$

Example 1.2.63 (Conservation of flow lines). A coframe $\theta = (\theta^1, \theta^2)$ determines a frame on $\mathbb{R}^2$ and this determines two vector fields (up to scalar multiples) on $\mathbb{R}^2$. The flow lines of these vector fields give a double foliation of $\mathbb{R}^2$. We want to know whether for every pair of foliations there exists a diffeomorphism mapping one foliation into the other.

Our problem is encoded by our coframe and the group $G = \mathrm{GL}(1, \mathbb{R}) \times \mathrm{GL}(1, \mathbb{R})$. Indeed, the foliation is determined by the span (over $C^\infty(\mathbb{R})$) of the vector fields determined by the coframe and the group G leaves these invariant. We parameterize the group by matrices $\mathrm{diag}(a, b)$, $ab \neq 0$. A basis for the left-invariant forms is $\alpha = a^{-1}\mathrm{d}a$, $\beta = b^{-1}\mathrm{d}b$, the structure equations for the lifted coframe after absorption of torsion are

$$\mathrm{d}\omega = -\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \wedge \omega.$$

At first sight it might appear that $s_1 = s_2 = 1$, but by taking different coordinates we see that $s_1 = 2$ and $s_2 = 0$. The dimension of the first Lie algebra prolongation is 2, Cartan's test is satisfied and all analytic double foliations are locally equivalent under analytic diffeomorphisms of the plane.

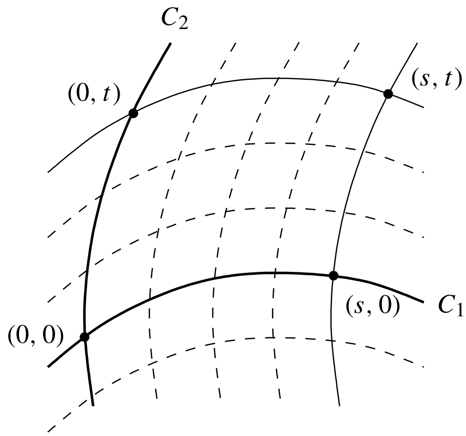

Figure 1.3: Construction of canonical coordinates for flow lines

This conclusion is also true in the C^∞ setting. At a special point x, y we can parameterize the curves C_1 and C_2 passing through this point with parameters s and t, respectively. Then we map the point (s, t) to the intersection of the flow curve that intersects C_1 at s and the flow curve that intersects C_2 at t. See Figure 1.3. This defines a local diffeomorphism that maps the flow lines to the standard flow lines defined by the coframe $\mathrm{d}x$, $\mathrm{d}y$. $\oslash$

1.3 Contact transformations

A *contact structure* on a manifold of dimension $(2n+1)$ is given by the kernel of a maximally non-degenerate 1-form α. The condition that α is maximally non-degenerate is

$$(\mathrm{d}\alpha)^n \wedge \alpha \neq 0.$$

A maximally non-degenerate 1-form is called a *contact form*. All contact structures are locally equivalent. This follows from the proposition below which is a consequence of Theorem 1.2.13.

Proposition 1.3.1. *Let α be a contact form. Then there are local coordinates z, $x^1, \ldots, x^n$, $p^1, \ldots, p^n$ such that*

$$\alpha = \mathrm{d}z - \sum_{j=1}^{n} p_j \mathrm{d}x^j.$$

The integral manifolds of maximal dimension of a contact structure $\mathcal{C}$ are integral manifolds of dimension n. These maximal integral manifolds are called *Legendre submanifolds* of $\mathcal{C}$, see Arnol′d et al. [6, pp. 312–316].

Example 1.3.2 (First order contact manifold). Let Z be a manifold of dimension $n+1$. Let P be the Grassmannian of n-planes in TZ. Let π be the projection $P \to Z$. The points in P are denoted as pairs $p = (z, E)$ with $z \in Z$ and $E \in \mathrm{Gr}_n(T_z Z)$. On P we define a contact structure $\mathcal{C}$ by

$$\mathcal{C}_p = (T_p \pi)^{-1}(E).$$

The pair $(P, \mathcal{C})$ is called the *first order contact manifold* of the *base manifold* Z. If $Z = \mathbb{R}^2 \times \mathbb{R}$, then the first order jet bundle $\mathrm{J}(\mathbb{R}^2, \mathbb{R})$ is an open subset of P. The contact structures on $\mathrm{J}(\mathbb{R}^2, \mathbb{R})$ and P are identical. $\oslash$

Let $(P, \mathcal{C})$ be a contact manifold of dimension $2n+1$. A *Legendre fibration* [6, p. 313] is a projection $\pi : P \to Z$ to a manifold Z of dimension $n+1$ such that π is a submersion and the fibers of $P \to Z$ are Legendre submanifolds of P. The manifold Z is called the base manifold of the fibration. The first order contact manifold of a manifold of dimension $n+1$ defines a Legendre fibration. Locally every Legendre fibration to a manifold of dimension $n+1$ is equivalent to the first order contact manifold of $\mathbb{R}^{n+1}$. This is proved in Duistermaat [24, Proposition 2.12].

1.3.1 Parameterization of infinitesimal contact transformations

Let P be a manifold of dimension $2n+1$ with a contact form α and dual plane field $\mathcal{C}$. A *contact transformation* is a diffeomorphism that preserves the contact structure. This implies that if α is a contact form for the contact structure, then any contact transformation ϕ satisfies $\phi^*\alpha \equiv 0 \mod \alpha$. An *infinitesimal contact transformation* is a vector field X such that $[X, Y] \subset \mathcal{C}$ for all $Y \subset \mathcal{C}$. In terms of the contact form α the vector field X is an infinitesimal contact transformation if $\mathcal{L}_X \alpha = g\alpha$ for an arbitrary function g. The (infinitesimal) contact transformations are also called (infinitesimal) symmetries of the contact structure. The 1-parameter subgroup of transformations defined by integration of an infinitesimal contact transformation defines a 1-parameter subgroup of contact transformations.

We will show that the infinitesimal contact transformations of a contact manifold of dimension $2n+1$ can be parameterized using one function of $2n+1$ variables. The idea for the construction below is from Kobayashi [46, Section I.7].

The quotient $TP/\mathcal{C}$ is a canonical line bundle over P. Since at all points $x \in P$ we have $\alpha(\mathcal{C}_x) = 0$ the contact form can be seen as a 1-form on $TP/\mathcal{C}$. There is a unique section s of $TP/\mathcal{C}$ such that $\alpha(s) = 1$. Now we can define

$$\tilde{\alpha} = s\alpha \in (TP/\mathcal{C}) \otimes \Omega^1(P)$$

as a 1-form on P with values in $TP/\mathcal{C}$. The form $\tilde{\alpha}$ is independent of the choice of representative α. The line bundle $TP/\mathcal{C} \to P$ with the form $\tilde{\alpha}$ also encodes the contact structure since $\ker \tilde{\alpha}_x = \mathcal{C}_x$.

The form $\tilde{\alpha}$ defines a map from the vector fields on P to sections of $TP/\mathcal{C}$ by $X \mapsto \tilde{\alpha}(X)$. We claim that this map is an isomorphism from the infinitesimal automorphisms of the contact structure to the sections of the line bundle $TP/\mathcal{C}$.

Proof. Let X be an infinitesimal symmetry for which $\tilde{\alpha}(X) = 0$. Then X must be contained in $\mathcal{C}$. Since X is a symmetry we must have $[X, Y] \subset \mathcal{C}$ for all $Y \subset \mathcal{C}$. But the Lie brackets modulo $\mathcal{C}$ are non-degenerate, hence $X = 0$. This proves the map is injective.

To prove that the map is surjective we let h be a section of $TP/\mathcal{C}$. We take a representative α of the contact structure and define S to be the unique vector field satisfying

$$\alpha(S) = 1, \quad S \lrcorner \mathrm{d}\alpha = 0.$$

By multiplying S with a suitable function ϕ we can arrange that $\tilde{\alpha}(\phi S) = h$. We want to construct an infinitesimal automorphism X such that $\tilde{\alpha}(X) = h$. We decompose X as $\phi S + Y$, with $Y \subset \mathcal{C}$. The condition that X is an infinitesimal symmetry is $[X, Z] \equiv 0 \mod \mathcal{C}$ for all $Z \in \mathcal{C}$. Hence

$$[Y, Z] \equiv Z(\phi)S \mod \mathcal{C}.$$

Since the Lie brackets are non-degenerate there is a unique element $Y \subset \mathcal{C}$ such that $Z \mapsto [Y, Z] \mod \mathcal{C}$ is equal to the linear map $\mathcal{C} \to TM/\mathcal{C} : Z \mapsto Z(\phi)S$. □

This implies there is a one-to-one correspondence between the infinitesimal contact symmetries and the sections of the canonical bundle $TP/\mathcal{C}$. After a local trivialization of $TP/\mathcal{C}$ these sections are given by a single function of $2n + 1$ variables.

Example 1.3.3. Consider the contact structure on $P = \mathrm{J}^1(\mathbb{R}^2) = \mathbb{R}^3$. We use coordinates x, z, p for P. The contact form is given by $\alpha = \mathrm{d}z - p\mathrm{d}x$. The dual contact distribution $\mathcal{C}$ is generated by $\partial_x + p\partial_z$ and ∂_p. We take $S = \partial_z$ as a representative element for the section s of $TP/\mathcal{C}$. Note that $\alpha(S) = 1$ and $S \lrcorner \mathrm{d}\alpha = 0$. From the general theory above we know that for every section ϕS of $TM/\mathcal{C}$ there is a unique infinitesimal symmetry $X = \phi S + Y$ of $\mathcal{C}$ such that $\tilde{\alpha}(X) = \phi S$. The vector field Y satisfies $(\mathrm{d}x \wedge \mathrm{d}p)(Y, Z) + \phi(Z) = 0$ for all $Z \in \mathcal{C}$. Using a basis for $\mathcal{C}$ we can calculate the vector field Y. We find that

$$X = \phi\partial_z - \phi_p(\partial_x + p\partial_z) + (\phi_x + p\phi_z)\partial_p.$$

The function ϕ is an arbitrary function of x, z and p.

Take for example $\phi = -p$. Then we find $X = \partial_x$ and this integrates to the contact transformation $(x, z, p) \mapsto (x + \epsilon, z, p)$. If we take $\phi = -(1/2)(p^2 + x^2)$, then we find the infinitesimal symmetry $X = p\partial_x - x\partial_p + (1/2)(p^2 - x^2)\partial_z$, which generates the *Legendre transformation* $(x, z, p) \mapsto (-p, z - xp, x)$. ⊘

1.3.2 Reeb vector fields and strict contact transformations

In some of the literature a contact structure is given by a contact form α on a manifold of dimension $2n+1$. The difference with our definition is that our contact structure is defined as the *kernel* of a contact form and contact forms which are proportional correspond to the same contact structure. We will distinguish these contact structures from our contact structures by calling them *strict contact structures*. From Proposition 1.3.1 it follows that all strict contact structures on $\mathbb{R}^3$ are locally equivalent. The symmetries of a strict contact structure on $\mathbb{R}^3$ depend on one function of two variables. This can be proved using the Cartan-Kähler theorem.

Given a contact form α there is a unique vector field R such that

$$\alpha(R) = 1, \quad R \lrcorner \mathrm{d}\alpha = 0. \tag{1.26}$$

This vector field is called the *Reeb vector field* for the contact form α.

For every Reeb vector field R we have

$$\mathcal{L}_R\alpha = R \lrcorner \mathrm{d}\alpha + \mathrm{d}(\alpha(R)) = 0 + \mathrm{d}(1) = 0.$$

Hence R is an infinitesimal symmetry of the contact form α and hence an infinitesimal contact symmetry of the contact structure $\mathcal{C}$. Conversely, let R be an infinitesimal contact symmetry of the contact structure defined by α that is pointwise not contained in $\mathcal{C}$. Then we can scale the contact form by a non-zero function such that $\alpha(R) = 1$. Since R is an infinitesimal contact symmetry we have $\mathcal{L}_R\alpha = f\alpha$ for a non-zero function f. On the other hand we have $\mathcal{L}_R\alpha = R \lrcorner \mathrm{d}\alpha = 0$ and hence $R \lrcorner \mathrm{d}\alpha = f\alpha$. By taking the interior product with R we find $f = 0$. So R is an infinitesimal symmetry of α and an infinitesimal contact symmetry.

Example 1.3.4. For a contact structure on $\mathbb{R}^5$ we can always introduce coordinates x, y, z, p, q such that the contact form is given by $\alpha = \phi(\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y)$, with ϕ a function of the coordinates x, y, z, p, q. The Reeb vector field is then given by

$$R = \phi^{-2}((\phi + p\phi_p + q\phi_q)\partial_z + \phi_p\partial_x + \phi_q\partial_y + (\phi_x + p\phi_z)\partial_p + (\phi_y + q\phi_z)\partial_q). \quad \oslash$$

1.3.3 Contact transformations and point transformations

Let P be the projectivized cotangent bundle of Z. We assume that Z has dimension $n + 1$. We say that the bundle $P \to Z$ is the first order contact manifold of Z, since the elements of P represent the first order contact of n-dimensional submanifolds of Z. The points in P consist of pairs (z, E) where $z \in Z$ and E is an n-dimensional linear subspace of $T_z Z$. Every transformation $\phi : Z \to Z$ lifts to a tansformation Φ on P that preserves the contact structure. The lift Φ is defined as $\Phi : P \to P : (z, E) \mapsto (\phi(z), T_z\phi E)$ and is called the prolongation of ϕ.

We consider infinitesimal contact transformations of P and are interested in those transformations that are not prolonged base transformations. Not all contact transformations are point transformations in local coordinates. For example consider the base manifold Z with coordinates x, y, z and the first order contact bundle P of Z with coordinates x, y, z, p, q. The point (x, y, z, p, q) corresponds to the linear subspace of $T_{(x,y,z)}Z$ spanned by the vectors $\partial_x + p\partial_z$ and $\partial_y + q\partial_z$. The Legendre transformation $\tilde{x} = p$, $\tilde{y} = q$, $\tilde{z} = z - px - qy$, $\tilde{p} = -x$, $\tilde{q} = -y$ is not a point transformation. On the other hand the infinitesimal contact transformation defined by the vector field $a\partial_p + b\partial_q$ for constants a, b is also not a point transformation, but after applying the Legendre transformation the vector field becomes $a\partial_{\tilde{x}} + b\partial_{\tilde{y}}$ which is the prolongation of an infinitesimal base transformation. We want to know which infinitesimal contact transformations are point transformations for a suitable choice of base coordinates.

Theorem 1.3.5. *Let $(P, \mathcal{C})$ be a contact manifold with infinitesimal contact transformation V. If $V_p \neq 0$, then in a neighborhood of $p \in P$ there exists a projection to a base manifold Z such that V is the prolongation of an infinitesimal base transformation in Z.*

Proof. The case $n = 0$ is trivial, so we may assume $n > 0$. At the point p choose a hypersurface H in P such that the tangent space of H at p is generic with respect to both $\mathcal{C}$ and V. The contact distribution $\mathcal{C}$ restricts on H to a codimension one bundle $\tilde{\mathcal{C}} = TH \cap \mathcal{C}$. Since the distribution $\tilde{\mathcal{C}}$ has rank $2n - 1$, corank 1 and is as non-degenerate as possible (this follows from the maximal non-degeneracy of $\mathcal{C}$), it has 1-dimensional Cauchy characteristics. Locally, the quotient of H under $C(\tilde{\mathcal{C}})$ is an $(2n - 1)$-dimensional contact manifold. In this contact manifold we can choose a foliation by integral curves. The integral curves lift to integral surfaces of the distribution $\tilde{\mathcal{C}}$ in H. This lifting is done by taking the flow of the integral curves by the Cauchy characteristics.

We have a hypersurface H that is foliated by integral surfaces. These integral surfaces are contained in $\mathcal{C}$ and hence they define Legendre submanifolds for $\mathcal{C}$. The vector field V was transversal to TH at p. Hence we can take the flow of H by V. This defines a local foliation of P by Legendre manifolds. The quotient of P by these Legendre manifolds is locally well-defined and defines a projection $\pi : P \to Z$. It is not difficult to check that Z is a base manifold for the contact manifold P. The vector field V preserves (by definition) the fibers of this projection and hence V projects to a vector field on Z. □

Corollary 1.3.6. *Any infinitesimal first order contact transformation V without a fixed point can be written in local coordinates as ∂_z.*

For infinitesimal contact transformations with fixed points, the construction above fails. In general for these contact transformations there need not exist a choice of coordinates such that the transformation is a prolonged base transformation. The author is not aware of any general theory here (such as obstructions to the existence of such coordinates in terms of the eigenvalues of the vector field), but we can give a counterexample to the previous theorem in the case of fixed points.

Example 1.3.7 (Legendre vector field). Consider the first order contact manifold of $\mathbb{R}^2$ with coordinates x, z, p and contact form $\alpha = \mathrm{d}z - p\mathrm{d}x$. We define the Legendre vector field V as

$$V = p\partial_x - x\partial_p + (1/2)(p^2 - x^2)\partial_z.$$

Since $\mathcal{L}_V(\alpha) = 0$ this vector field generates a 1-parameter family of contact transformations. Explicit integration yields

$$\begin{aligned} x(t) &= x_0 \cos(t) + y_0 \sin(t), \\ p(t) &= y_0 \cos(t) - x_0 \sin(t), \\ z(t) &= z_0 + (1/2)x_0 y_0 \cos(2t) - (1/4)x_0^2 \sin(2t) + (1/4)y_0^2 \sin(2t). \end{aligned}$$

For $t = \pi/2$ we find the Legendre transformation $\tilde{x} = p, \tilde{z} = z - px, \tilde{p} = -x$. Every foliation that is invariant under the flow of V must have the z-axis as an element of the foliation. But the z-axis does not define a Legendre manifold since α does not restrict to zero. Therefore for any choice of base coordinates, i.e., a choice of foliation of the contact manifold by Legendre manifolds, the vector field V will not preserve the leaves of the foliation through the z-axis. We conclude that V cannot be an infinitesimal point transformation. $\oslash$

Chapter 2

Surfaces in the Grassmannian

In this chapter we study hyperbolic surfaces in the Grassmannian of 2-planes in a 4-dimensional vector space V. This type of surface occurs naturally in the study of partial differential equations. See the beginning of Section 2.3 for the relation between these surfaces and partial differential equations. The elliptic surfaces have already been described by McKay [51, 52] using complex numbers.

For hyperbolic surfaces we will define the equivalent of complex numbers: the hyperbolic numbers. The hyperbolic numbers form an algebra with properties very similar to the complex numbers and provide a convenient way to organize the calculations. For the compact hyperbolic surfaces we obtain a topological classification, like the classification for compact elliptic surfaces described in Gluck and Warner [39]. A compact hyperbolic surface is either a torus or a Klein bottle. We also study a special class of hyperbolic surfaces called the geometrically flat surfaces. We show that, even though the condition for a surface to be geometrically flat is quite rigid, there exist several different classes of geometrically flat surfaces.

We conclude the study by giving a calculation of the local invariants of hyperbolic surfaces under the action of the general linear transformations of the vector space V. Because the group acting is finite-dimensional, we can give a complete description of the invariants at all orders. We also give a geometric construction of the invariants similar to the construction given in McKay [51, pp. 25–30]. This geometric construction of the invariants will be used in Section 5.3 to make a connection to the invariants of first order systems.

2.1 Grassmannians

Let V be an n-dimensional vector space. The *Grassmannian* $\mathrm{Gr}_k(V)$ is defined as the set of all k-dimensional linear subspaces of V. The k-dimensional linear subspaces of V are also called *k-planes* in V. The group $\mathrm{GL}(V)$ acts transitively on V and this induces a transitive action on $\mathrm{Gr}_k(V)$. The stabilizer group of a k-plane L is the group $H = \{\, g \in \mathrm{GL}(V) \mid g(L) = L \,\}$. The Grassmannian is a homogeneous space G/H of dimension $k(n-k)$.

Given an element $L \in \mathrm{Gr}_k(V)$ we can introduce local coordinates for $\mathrm{Gr}_k(V)$ in the following way. Select a complementary subspace M such that $L \oplus M = V$. Let $\mathrm{Gr}_k^0(V, M)$ be the open subset of $\mathrm{Gr}_k(V)$ of k-planes that have zero intersection with M.

Lemma 2.1.1. *The space* $\mathrm{Lin}(L, M)$ *is diffeomorphic to* $\mathrm{Gr}_k^0(V, M)$ *through the map*

$$A \in \mathrm{Lin}(L, M) \mapsto \{ x + Ax \mid x \in L \} \in \mathrm{Gr}_k(V).$$

The diffeomorphisms described in the previous lemma for different k-planes L, M provide coordinate charts for $\mathrm{Gr}_k(V)$. The coordinate transformations between these charts are rational maps.

Sometimes we are interested in the oriented k-planes. The stabilizer group $\tilde{H}$ of the oriented k-planes is smaller than the stabilizer group H of the unoriented k-planes. The manifold $\widetilde{\mathrm{Gr}}_k(V)$ of oriented k-planes is equal to the quotient $G/\tilde{H}$. Locally $\mathrm{Gr}_k(V)$ and $\widetilde{\mathrm{Gr}}_k(V)$ are diffeomorphic. The space of oriented k-planes is a 2-fold cover of the space of unoriented k-planes.

In the case of 2-planes in $V = \mathbb{R}^4$ there is another view of the Grassmannian. Every 2-plane can be represented by 2 linearly independent vectors v, w. Such a pair defines an element $v \wedge w$ of $\Lambda^2(V)$. Since $\Lambda^4(V) \cong \mathbb{R}$ the map

$$\lambda : \Lambda^2(V) \to \Lambda^4(V) : x \mapsto x \wedge x$$

can be viewed as a homogeneous polynomial of degree 2. The elements $v \wedge w$ that represent a 2-plane all satisfy $\lambda(v \wedge w) = v \wedge w \wedge v \wedge w = 0$. Conversely, if an element $x \in \Lambda^2(V) \setminus \{0\}$ satisfies $\lambda(x) = 0$, then it can be written as $x = v \wedge w$ for two linearly independent vectors $v, w \in V$.

Lemma 2.1.2. *The Grassmannian of* 2*-planes in a 4-dimensional vector space V is isomorphic to*

$$N = \{ x \in \Lambda^2(V) \mid x \neq 0, \lambda(x) = x \wedge x = 0 \}/\mathbb{R}^* \subset \mathbb{P}(\Lambda^2(V)).$$

Let $\tilde{N} = \{ x \in \Lambda^2(V) \mid x \neq 0,\ x \wedge x = 0 \}$. So the non-zero elements of $\tilde{N}$ modulo a scalar represent 2-planes. For a point $x \in N$ the tangent space $x + T_x N$ to N at x is a linear subspace of $\Lambda^2 V$. The zero set of λ defines a smooth quadratic hypersurface in $\mathbb{P}(\Lambda^2(V))$. The description of the Grassmannian as a smooth quadratic in $\mathbb{P}^5$ is due to Plücker [60]. The oriented Grassmannian is isomorphic to the quadratic defined by λ in $\Lambda^2(V)/\mathbb{R}^+$.

2.1.1 Conformal quadratic form

Lemma 2.1.3. *Let P be a k-plane in* $\mathrm{Gr}_k(\mathbb{R}^n)$*. Then* $T_P\,\mathrm{Gr}_k(\mathbb{R}^n)$ *is canonically isomorphic to* $\mathrm{Lin}(P, \mathbb{R}^n/P)$*.*

Proof. We introduce local coordinates near P by choosing a complementary $(n-k)$-plane Q. The linear maps $\mathrm{Lin}(P, Q)$ are isomorphic with a neighborhood of P in $\mathrm{Gr}_k(\mathbb{R}^n)$. Since Q was chosen transversal to P, there is a natural isomorphism from Q to $\mathbb{R}^n/P$. The tangent space of $\operatorname{Lin}(P, Q)$ is just $\operatorname{Lin}(P, Q)$ and hence we find an identification of $T_P \operatorname{Gr}_k(\mathbb{R}^n)$ with $\operatorname{Lin}(P, \mathbb{R}^n/P)$, depending on a choice of a transversal plane Q.

We will show that the identification of $T_P \operatorname{Gr}_k(\mathbb{R}^n)$ with $\operatorname{Lin}(P, \mathbb{R}^n/P)$ does not depend on the choice of a transversal plane Q. Let Q' be another transversal plane. Then Q' is of the form $\{ x + Bx \mid x \in Q \}$ for a linear map $B : Q \to P$. This also gives the identification of Q, Q' and $\mathbb{R}^n/P$. A tangent vector at P is represented by a curve $X(t)$ in $\operatorname{Lin}(P, Q)$ with $X(0) = 0$. The elements on the curve are k-planes defined by $\{ z + X(t)z \mid z \in P \}$. We can rewrite the curve as

$$z + X(t)z = (I - BX(t))\, z + (X(t) + BX(t))\, z. \tag{2.1}$$

Note that for small t the map $I - BX(t)$ is a linear isomorphism $P \to P$. For $X(t)$ near zero we can make a different parameterization with $(I - BX(t))z = y$. Then the curve is represented by a map $P \to Q'$ given by

$$(X(t) + BX(t))(I - BX(t))^{-1}.$$

The curve in the Grassmannian is given by $y + (X(t) + BX(t))(I - BX(t))^{-1}y$. Differentiating the both representations gives $X'(t)$ for the first representation and $X'(t) + BX'(t)$ for the representation (2.1). But as representative elements for $\operatorname{Lin}(P, \mathbb{R}^n/P)$ the forms $X'(t)$ and $X'(t) + BX'(t)$ are equal, since $BX'(t) \in P$. □

On the tangent space of the Grassmannian of 2-planes in $\mathbb{R}^4$ there is a conformally invariant quadratic form. In the case that $n = 4$ and $k = 2$, we can identify $\operatorname{Lin}(P, \mathbb{R}^4/P)$ after a choice of basis in P and $\mathbb{R}^4/P$ with the space of 2×2-matrices. The determinant of a 2×2-matrix defines a quadratic form of signature $(2, 2)$. This gives a quadratic form on the tangent space of $\operatorname{Gr}_2(\mathbb{R}^4)$ that depends on the choice of basis. Modulo a scalar factor this quadratic form is well-defined and hence we have an invariant conformal quadratic form ξ on the tangent space. For other introductions to this conformal quadratic form see Akivis and Goldberg [1, pp. 19–23] or McKay [51, pp. 19–20].

Remark 2.1.4. Another way to define the conformal quadratic form is to use the representation of the Grassmannian as $N \subset \mathbb{P}(\Lambda^2(V))$. The conformal quadratic form $\lambda : \mu \mapsto \mu \wedge \mu$ defining N is trivial on N, but since N is a not a linear space this map is not trivial on the tangent space. Write $\mu(t) = \mu_0 + t\dot{\mu} + \mathcal{O}(t^2)$ for a curve in N. Then the conformal quadratic form on the tangent space is given by $\xi : \dot{\mu} \mapsto \dot{\mu} \wedge \dot{\mu}$.

We should warn the reader not the confuse the two forms: λ is a conformal quadratic form on $\mathbb{P}(\Lambda^2(V))$ that defines the Grassmannian, ξ is a conformal quadratic form on the tangent space of the Grassmannian. ⊘

The group $\mathbb{P}\operatorname{GL}(4, \mathbb{R})$ acts on $\mathbb{P}V$ and this induces a transformation of $\operatorname{Gr}_2(V)$. The conformal isometries induced from the action of $\mathbb{P}\operatorname{GL}(V)$ are given in local coordinates by

$$A \mapsto (c + dA)(a + bA)^{-1}. \tag{2.2}$$

These transformations are also called *Möbius transformations*. In the local coordinates given by 2×2-matrices A for the Grassmannian, the conformal quadratic form on the tangent space is given by $A \mapsto \det(A)$. From the form in local coordinates we see that the conformal structure on the Grassmannian is equivalent to the flat conformal structure on $\mathbb{R}^4$.

Lemma 2.1.5. *The conformal isometry group of* $\mathrm{Gr}_2(V)$ *is equal to* $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$. *The group* $\mathbb{P}\,\mathrm{GL}^+(4, \mathbb{R})$ *of orientation preserving projective linear transformations is the conformal isometry group of* $\widetilde{\mathrm{Gr}}_2(V)$.

Proof. Every element of $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$ induces a transformation of the form (2.2). These transformations are conformal and the action of $A \in \mathrm{GL}(4, \mathbb{R})$ is non-trivial on the Grassmannian if and only if $A \in \mathbb{R}I$. In Section A.5.4 we proof that every conformal transformation can be obtained from $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$. □

A 2-dimensional linear subspace of the tangent space $T\,\mathrm{Gr}_2(\mathbb{R}^4)$ will be called a *tangent 2-plane* or just a tangent plane. For every tangent 2-plane E the conformal quadratic form on $\mathrm{Gr}_2(\mathbb{R}^4)$ restricts to a conformal quadratic form on E.

Definition 2.1.6. A tangent 2-plane in $T\,\mathrm{Gr}_2(\mathbb{R}^4)$ is called *elliptic* if the conformal quadratic form restricts to a positive or negative definite non-degenerate quadratic form. A tangent 2-plane in $T\,\mathrm{Gr}_2(\mathbb{R}^4)$ is called *hyperbolic* if the conformal quadratic form restricts to a non-degenerate quadratic form of signature $(1, 1)$. ⊖

A quadratic form ξ on a vector space W is equivalent to a symmetric bilinear form on W. Any conformal quadratic form defines an *isotropic cone* $C = \{\, w \in W \otimes \mathbb{C} \mid \xi(w) = 0 \,\}$. If W is 2-dimensional and the conformal quadratic form is non-degenerate, then the isotropic cone consists of two distinct complex one-dimensional linear subspaces which are called the *characteristic lines* of the conformal quadratic form. If the conformal quadratic form is definite, then the intersection of the isotropic cone with W consists of the origin. If the form is indefinite, then the intersection of the isotropic cone with W consists of two 1-dimensional lines in W. We call these lines the characteristic lines as well.

Theorem 2.1.7. *The general linear group* $\mathrm{GL}(4, \mathbb{R})$ *acts transitively on the Grassmannian of* 2*-planes. At each point in the Grassmannian the stabilizer subgroup of that point acts transitively on the elliptic tangent planes and also transitively on the hyperbolic tangent planes.*

Proof. Since the Grassmannian was realized as a homogeneous space it is clear that $\mathrm{GL}(4, \mathbb{R})$ acts transitively. The action of the stabilizer group is analyzed in Appendix A.5.3. The orbits of the elliptic and hyperbolic 2-planes are the only two open orbits in the tangent space to the Grassmannian. □

2.1.2 Plücker coordinates

We have described the Grassmannian $\mathrm{Gr}_2(V)$ as the space of elements q in $\Lambda^2(V)$ that satisfy $q \wedge q = 0$ modulo a scalar factor. In this section we will use the eigenspaces of the Hodge $*$ operator to further explain the structure of the Grassmannian $\mathrm{Gr}_2(V)$ and the conformal quadratic form ξ.

Let e_1, e_2, e_3, e_4 form a basis for V. With respect to the volume form $\Omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ we have the Hodge star operator $* : \Lambda^2(V) \to \Lambda^2(V)$. The operator depends on the choice of basis and the choice of the volume form Ω, but since the Grassmannian is defined as a subspace of the projective space $\mathbb{P}\Lambda^2(V)$ the choice of volume form is not essential.

We define

$$\begin{aligned} \alpha_1 &= (1/2)(e_1 \wedge e_2 + e_3 \wedge e_4), & \beta_1 &= (1/2)(e_1 \wedge e_2 - e_3 \wedge e_4),\\ \alpha_2 &= (1/2)(e_1 \wedge e_3 - e_2 \wedge e_4), & \beta_2 &= (1/2)(e_1 \wedge e_3 + e_2 \wedge e_4),\\ \alpha_3 &= (1/2)(e_1 \wedge e_4 + e_2 \wedge e_3). & \beta_3 &= (1/2)(e_1 \wedge e_4 - e_2 \wedge e_3). \end{aligned} \tag{2.3}$$

The forms α_i, β_j satisfy

$$\alpha_i \wedge \beta_j = 0, \quad \alpha_i \wedge \alpha_j = \delta_{ij}\Omega, \quad \beta_i \wedge \beta_j = -\delta_{ij}\Omega.$$

The eigenspaces of the Hodge operator are $E_+ = \langle \alpha^j \rangle$, $E_- = \langle \beta^j \rangle$ corresponding to the eigenvalues 1 and -1 of $*$, respectively. We can decompose any $\eta \in \Lambda^2(V)$ in terms of these eigenspaces. Write $\eta = X^i \alpha_i + Y^j \beta_j$. The coefficients X^i, Y^j can be used to parameterize the Grassmannian and are called *Plücker coordinates*. The name Plücker coordinates is misleading because the coefficients do not define real coordinates for $\mathrm{Gr}_2(V)$. A pair (X, Y) only defines an element of the Grassmannian if the Plücker form λ is zero and two elements that are a scalar multiple of each other define the same element in the Grassmannian.

The conformal quadratic form λ acts on η as

$$\begin{aligned} \lambda(\eta) = \eta \wedge \eta &= (X^i \alpha_i + Y^j \beta_j) \wedge (X^i \alpha_i + Y^j \beta_j)\\ &= X^i X^j \alpha_i \wedge \alpha_j + Y^i Y^j \beta_i \wedge \beta_j = |X|^2 - |Y|^2. \end{aligned}$$

Lemma 2.1.8. *Let S^+ and S^- be two copies of the 2-sphere $S^2 \subset \mathbb{R}^3$. Then the map*

$$S^+ \times S^- \to \Lambda^2(V)/\mathbb{R}^+ : (X, Y) \mapsto X^i \alpha_j + Y^j \beta_j$$

defines a diffeomorphism from $S^+ \times S^-$ to the oriented Grassmannian.

This result is from Gluck and Warner [39]. Since $(X, Y) \in S^+ \times S^-$ satisfies $|X|^2 = |Y|^2 = 1$, the image of this map is contained in N. Is is not difficult to see that the map defines an isomorphism from $S^+ \times S^-$ to N.

2.1.3 Incidence relations

Let V be a vector space of dimension n and let L_0 be a point in $\mathrm{Gr}_k(V)$. We define $\Sigma_{L_0} = \{ L \in \mathrm{Gr}_k(V) \mid L \cap L_0 \neq 0 \}$. Locally we can describe the set Σ_{L_0} as the subset of $(n-k) \times k$ matrices which have non-trivial kernel. If we choose a transversal $(n-k)$-plane M and use the local coordinates from Lemma 2.1.1, then $\Sigma_{L_0} \cap \mathrm{Gr}_k^0(V, M) = \{ A \in \mathrm{Lin}(L_0, M) \mid \ker A \neq 0 \}$. If $n = 2k$, then Σ_{L_0} is determined by the $k \times k$-matrices with determinant zero. This is a hypersurface in the Grassmannian with a conical singularity at the zero matrix.

We consider the special case $n = 2k = 4$. Here we have only a singularity at L_0 itself. Consider the map

$$\Sigma_{L_0} \setminus \{ L_0 \} \to \mathrm{Gr}_1(L_0) : L \mapsto L \cap L_0.$$

The map is surjective and the fiber above a point $l \in \mathbb{P}^1(L_0)$ is equal to $I_l = \{ L \in \mathrm{Gr}_2(V) \mid l \subset L,\ L \neq L_0 \}$. The set I_l is equal to $\mathrm{Gr}_2(V/l)$ minus the point L_0, so this is a plane $\mathbb{R}^2$.

2.2 Hyperbolic theory

2.2.1 Hyperbolic numbers

Hyperbolic numbers are closely related to complex numbers. In the next chapters we will use them to write structure equations in a compact way and to organize the calculations.

Definition 2.2.1. On $\mathbb{R}^2$ we define a multiplication by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}. \tag{2.4}$$

The set $\mathbb{R}^2$ with this multiplication is an algebra $\mathbb{D}$ over $\mathbb{R}$, which we call the *hyperbolic numbers*. Multiplication of a hyperbolic number with a scalar is given by

$$\mathbb{R} \times \mathbb{D} \to \mathbb{D} : \left(\lambda, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}. \qquad \ominus$$

A hyperbolic number x for which $x_1 = x_2$ is called *real hyperbolic*. A hyperbolic number for which $x_1 = -x_2$ is called *imaginary hyperbolic*, or imaginary for short. We write h for the special number $(1, -1)^T$. We can then write every hyperbolic number x as $x = a + h \cdot b$, with $a, b \in \mathbb{R}$. The number h has the property $h^2 = 1 \in \mathbb{D}$.

The algebra $\mathbb{D}$ is isomorphic to the ring $\mathbb{R}[X]/(X^2 - 1)$ of polynomials in the variable X modulo the ideal generated by $X^2 - 1$. The map $a + hb \mapsto a + bX$ is an algebra isomorphism. The complex numbers are isomorphic to $\mathbb{R}[X]/(X^2 + 1)$. This shows that there is close relation between $\mathbb{D}$ and the complex numbers. The algebra $\mathbb{D}$ is also known as the *split-complex numbers*, double numbers or countercomplex numbers. See Wikipedia [74] for a complete list with references.

Definition 2.2.2. Let $x = (x_1, x_2)^T$ be a hyperbolic number. The *flip* of x is defined as

$$x^F = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}. \qquad \ominus$$

The flip operation is the equivalent of the conjugation on complex numbers. Note that for all hyperbolic numbers $(a + hb)^F = a - hb$ and $(x^F)^F = x$, so the flip operation is indeed very similar to the conjugation of complex numbers. The ring of hyperbolic numbers has zero divisors. For example $(1 + h)(1 - h) = 1 - h^2 = 0$. We will write $\mathbb{D}^*$ for the group of invertible hyperbolic numbers. We define the square of the norm of a hyperbolic number by $|x|^2 = xx^F$. The element $|x|^2$ is real hyperbolic, so we can treat it as a real number. The value of $|x|^2$ can be negative, so using the name norm for $|a|$ can be a bit misleading.

Matrix representation of hyperbolic numbers

In the following it is useful to have several different representations of hyperbolic numbers. The first one is the representation as elements of $\mathbb{R}^2$. In this section we will represent the hyperbolic numbers as a suitable subgroup of the algebra of 2×2-matrices.

We write D for the space of 2×2-diagonal matrices. Note that D is a commutative algebra. With a hyperbolic number x, i.e., a pair of variables x_1, x_2, we can associate a diagonal 2×2-matrix $x = (x_1, x_2)$. This defines an isomorphism

$$\mathbb{D} \to D : \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \mapsto \begin{pmatrix} x^1 & 0 \\ 0 & x^2 \end{pmatrix}.$$

On D we have an involution which maps x to the diagonal matrix $x^F = (x_2, x_1)$. The involution is an algebra homomorphism, i.e., for all $x, y \in D$ we have $(x + y)^F = x^F + y^F$ and $(xy)^F = x^F y^F$. Define L to be the matrix

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.5}$$

Then the flip operation can also be written in terms of matrix multiplication $x^F = LxL^{-1}$.

Functions

The usual operations from algebra and differential geometry can be applied to $\mathbb{D}$-valued functions and differential forms on a manifold M. For example the exterior differential operator d can be applied to a 1-form $\omega = (\omega^1, \omega^2)^T \in \mathbb{D} \otimes \Omega(M)$ as

$$\mathrm{d}\omega = \begin{pmatrix} \mathrm{d}\omega^1 \\ \mathrm{d}\omega^2 \end{pmatrix} \in \mathbb{D} \otimes \Omega^2(M).$$

The precise meaning of constructions with elements from $\mathbb{D}$ will usually be clear from the context. If we have a hyperbolic variable z and a function w of z that takes values in $\mathbb{D}$, then we define $\partial w/\partial z^F = (\partial w^1/\partial z^2, \partial w^2/\partial z^1)^T$. We say the function w is *hyperbolic holomorphic* if $\partial w/\partial z^F = 0$.

2.2.2 Hyperbolic structures

A hyperbolic structure is the analogue of an almost complex structure, see Chern et al. [20, Chapter 7] or Kobayashi and Nomizu [48, Chapter IX]. Some authors also use the term (almost) product structure.

Definition 2.2.3 (Hyperbolic structure). Let V be a $2n$-dimensional real vector space. A *hyperbolic structure* on V is an endomorphism $K : V \to V$ such that $K^2 = I$ and the eigenvalues ± 1 of K both occur with geometric multiplicity n. ⊖

The condition that $K^2 = I$ implies that the eigenspaces $V_\pm$ of K for the eigenvalues ± 1 span V. Hence the algebraic multiplicity will always equal the geometric multiplicity.

Definition 2.2.4 (Alternative definition of hyperbolic structure). Let V be a $2n$-dimensional vector space. A *hyperbolic structure* on V is a direct sum decomposition $V = V_+ \oplus V_-$ into a positive and negative part with $\dim V_+ = \dim V_- = n$.

Proof. Given a direct sum decomposition of V we can define an endomorphism $K : V \to V$ by requiring K to be the identity on V_+ and minus the identity on V_-. It is easy to check that K defines a hyperbolic structure. Conversely, given a hyperbolic structure on V we can define $V_\pm$ to be the eigenspace corresponding to the eigenvalue ± 1. □

If V is a vector space with hyperbolic structure K, then we can multiply the vectors in V by hyperbolic numbers. For $x = a + hb$ a hyperbolic number and $Y \in V$ we define the scalar multiplication by

$$\mathbb{D} \times V \to V : (x, Y) \mapsto aY + bKY.$$

This turns V into a module over $\mathbb{D}$.

For a vector X we denote by $X_+ + X_-$ the decomposition of X into eigenvectors for the hyperbolic structure. We say a vector is *generic* with respect to the hyperbolic structure if $X_+ \neq 0$ and $X_- \neq 0$. Another equivalent definition of a generic vector is that the vectors X and KX are linearly independent in V.

Example 2.2.5 (Involutions). Let $V = \mathbb{R}^4$ and consider the following three linear maps

$$K_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$K_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

All maps K_1, K_2, K_3 have eigenvalues ± 1, but only K_3 defines a hyperbolic structure. The matrix K_1 does not have the proper multiplicities for the eigenvalues and K_2 has a nilpotent part so that $(K_2)^2 \neq I$. ⊘

Example 2.2.6 (Standard hyperbolic structure). Let $V = \mathbb{R}^{2n}$. The standard hyperbolic structure on V is given by the diagonal matrix K with entries $K_{ii} = (-1)^{i+1}$. The standard hyperbolic structure on $\mathbb{R}^2$ is given by

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ ⊘

Remark 2.2.7. A hyperbolic structure on an even dimensional vector space does not determine an orientation. This is in contrast with a complex structure that does determine an orientation. Consider for example the standard hyperbolic structure on $\mathbb{R}^4$ given by

$$K_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The linear map given by the diagonal matrix $A = (1, -1, 1, 1)$ preserves the hyperbolic structure, i.e., $AK = KA$, but is orientation reversing. $\oslash$

2.2.3 Almost product manifolds

An *almost product structure* K on a manifold M is a hyperbolic structure on the tangent space at every point of the manifold. The pair (M, K) is called an *almost product manifold* or *hyperbolic manifold*. For all points x in M the almost product structure K gives $T_x M$ the structure of a $\mathbb{D}$-module. Given an almost product manifold M with hyperbolic structure $K : TM \to TM$ we can define the *hyperbolic Nijenhuis tensor*.

Definition 2.2.8. Let (M, K) be an almost product structure. The map

$$\begin{aligned} N = [K, K] : TM \times_M TM &\to TM \\ (X, Y) &\mapsto [KX, KY] - K[KX, Y] \\ &\quad - K[X, KY] + K^2[X, Y]. \end{aligned} \tag{2.6}$$

$\ominus$

is a tensor. We call this tensor the (hyperbolic) *Nijenhuis tensor*.

The Nijenhuis tensor is defined for general endomorphisms $K : TM \to TM$. See Kobayashi and Nomizu [48, p. 123]. The cases $K^2 = 1$ and $K^2 = -1$ are the most interesting.

Just like the Nijenhuis tensor for almost complex structures, the hyperbolic Nijenhuis tensor is anti-symmetric. It also is hyperbolic anti-linear in both variables, i.e., for $X, Y \in TM$ and $a, b \in \mathbb{D}$ we have

$$N(aX, bY) = a^F b^F N(X, Y).$$

The vanishing of the Nijenhuis tensor in the hyperbolic case is equivalent to a (local) direct product structure of the bundle, in the smooth category. Suppose that the hyperbolic structure on a manifold M is given by $K : TM \to TM$. Since K is a hyperbolic structure we know that $K^2 = I$ and the multiplicity of the eigenvalues $+1$ and -1 is the same.

Let the distribution $\mathcal{F}$ be given by all vectors of the form $X + KX$ and the distribution $\mathcal{G}$ by the vectors of the form $Y + KY$. So $\mathcal{F}$ corresponds to the eigenspace of K for eigenvalue 1 and $\mathcal{G}$ to the eigenspace of K for eigenvalue -1. Then the algebraic properties of K imply that $\mathcal{F}$ and $\mathcal{G}$ are smooth vector subbundles of constant rank. We prove that $\mathcal{F}$ is integrable.

Let $X' = (X + KX) \subset \mathcal{F}$ and $Y' = (Y + KY) \subset \mathcal{F}$ be two arbitrary vector fields in the bundle $\mathcal{F}$. The Lie bracket of X and Y is given by

$$Z = [X + KX, Y + KY] = [X, Y] + [X, KY] + [KX, Y] + [KX, KY].$$

The condition that $Z \subset \mathcal{F}$ is given by

$$\begin{aligned} 0 = Z - KZ &= [X, Y] + [X, KY] + [KX, Y] + [KX, KY] \\ &\quad - K([X, Y] + [X, KY] + [KX, Y] + [KX, KY]) \\ &= [KX, KY] - K[X, KY] - K[KX, Y] + [X, Y] \\ &\quad - K([KX, KY] - K[X, KY] - K[KX, Y] + [X, Y]) \\ &= N(X, Y). \end{aligned}$$

The vanishing of the Nijenhuis tensor is equivalent to both $\mathcal{F}$ and $\mathcal{G}$ being integrable. If both $\mathcal{F}$ and $\mathcal{G}$ are integrable, then this defines a decomposition of M into a direct product manifold with $\mathcal{F}$ and $\mathcal{G}$ equal to the tangent spaces of the components. If an almost product manifold has vanishing Nijenhuis tensor we say the structure is *integrable* and the manifold is a direct product manifold.

Example 2.2.9 (Surfaces with an almost product structure). Consider a manifold of dimension two with an almost product structure. The almost product structure defines two rank one distributions on the surface. The integral curves of these distributions are called the *characteristic curves* for the surface. Since the Nijenhuis tensor is anti-symmetric it must vanish and hence the surface is (locally) a direct product manifold. The tangent spaces of the components are the eigenspaces of the almost product structure K. The tangent spaces are also equal to the distributions mentioned above. Just as complex surfaces, the surfaces with an almost product structure have no local invariants. Locally all these surfaces are equivalent.⊘

Let M be an almost product manifold with hyperbolic structure $K : TM \to TM$. Recall that $h \in \mathbb{D}$ is the hyperbolic number $(1, -1)^T$. A *hyperbolic holomorphic* function on M is a $\mathbb{D}$-valued function f on M that satisfies $h \circ (\mathrm{d}f) = (\mathrm{d}f) \circ K$. A surface in M is called a *hyperbolic pseudoholomorphic curve* if the tangent space to the surface is K-invariant and the eigenspaces of K have non-zero intersection with the tangent space to the surface. The notions of a hyperbolic holomorphic function and a hyperbolic pseudoholomorphic curve are the equivalents to holomorphic functions and pseudoholomorphic curves for almost complex structures.

2.2.4 Hyperbolic groups

The reader probably is already familiar with the fact that the complex groups $\mathrm{GL}(n, \mathbb{C})$ can be embedded in $\mathrm{GL}(2n, \mathbb{R})$. We will show that in the same way we can embed the hyperbolic groups $\mathrm{GL}(n, \mathbb{D})$ into the general linear group $\mathrm{GL}(2n, \mathbb{R})$. The $\mathbb{D}$-linear endomorphisms $\mathrm{Lin}(\mathbb{D}^n, \mathbb{D}^n)$ from $\mathbb{D}^n$ to $\mathbb{D}^n$ can be identified with the $n \times n$-matrices with entries in $\mathbb{D}$. The maps that are invertible are the $\mathbb{D}$-linear automorphisms, which we denote by $\mathrm{GL}(n, \mathbb{D})$. The group $\mathrm{GL}(n, \mathbb{D})$ is the set of elements $x \in \mathrm{Lin}(\mathbb{D}^n, \mathbb{D}^n)$ that are invertible. The invertible elements x are precisely the elements which have an invertible determinant.

To each hyperbolic number we can associate a 2×2-matrix, see Section 2.2.1. For every element x in $\mathrm{Lin}(\mathbb{D}^n, \mathbb{D}^n)$ we can define an element of $\mathrm{Lin}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ by replacing the entries of x by the corresponding 2×2-matrices. This gives an embedding of $\mathrm{GL}(n, \mathbb{D})$ in $\mathrm{GL}(2n, \mathbb{R})$.

Example 2.2.10. Let $x \in \mathrm{Lin}(\mathbb{D}^2, \mathbb{D}^2)$ be given by

$$\begin{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} & \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \\ \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} & \begin{pmatrix} a_4 \\ b_4 \end{pmatrix} \end{pmatrix}.$$

This element is mapped to the 4×4-matrix

$$\begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & b_1 & 0 & b_2 \\ a_3 & 0 & a_4 & 0 \\ 0 & b_3 & 0 & b_4 \end{pmatrix} \in \mathrm{Lin}(\mathbb{R}^4, \mathbb{R}^4). \qquad \oslash$$

Decomposition

Let V be $\mathbb{R}^{2n}$ with the standard hyperbolic structure K, see Example 2.2.6. There is a unique decomposition of elements $\tilde{A} \in \mathrm{Lin}_K(V, V)$ into a K-linear and K-antilinear part. We can write any $2n \times 2n$ matrix as $\tilde{A} = A' + B$ with

$$A' = (\tilde{A} + K\tilde{A}K)/2, \quad B = (\tilde{A} - K\tilde{A}K)/2.$$

Note that $KA' = A'K$ and $KB = -BK$. The decomposition of $\tilde{A}$ into A' and B can be compared to the decomposition of a matrix into its complex-linear and complex-antilinear part.

For any $L \in \mathrm{GL}(V)$ for which $L^2 = I$ and $KL = -LK$ the map $X \mapsto XL$ is an isomorphism from the K-linear matrices to the K-antilinear matrices. This allows us to identify the K-linear and K-antilinear matrices. In the case $n = 2$ we take L to be the matrix given in (2.5). We can write any $2n \times 2n$ matrix as $\tilde{A} = A' + A''L$ with

$$A' = (\tilde{A} + K\tilde{A}K)/2, \quad A'' = BL = ((\tilde{A} - K\tilde{A}K)/2)L.$$

The matrix A' is called the *hyperbolic part* of $\tilde{A}$ and $A''L = (\tilde{A} - K\tilde{A}K)/2$ is called the *anti-hyperbolic part* of $\tilde{A}$ (with respect to K). By abuse of language we will also call A'' the *anti-hyperbolic part* of $\tilde{A}$. The elements A' and A'' can be regarded as elements from $\mathrm{Lin}(\mathbb{D}^n, \mathbb{D}^n)$ through the identification described in the previous section.

Example 2.2.11. Let

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Lin}(\mathbb{R}^2, \mathbb{R}^2).$$

Then for the standard hyperbolic structure we have

$$A' = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad A'' = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} L = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}.$$

A vector $x = (x^1, x^2)^T$ multiplied by the 2×2-matrix $\tilde{A}$ can be written as

$$\tilde{A}x = A'x + A''x^F = \begin{pmatrix} ax_1 + bx_2 \\ dx_2 + cx_1 \end{pmatrix}. \qquad \oslash$$

2.3 Microlocal analysis

Let S be a surface in the Grassmannian. The conformal quadratic form on the tangent space of the Grassmannian restricts to a quadratic form on the tangent space of S. For generic tangent spaces the form is non-degenerate and is either definite or indefinite. If the conformal quadratic form is definite this defines an almost complex structure on the surface and if the form is indefinite this defines an almost product structure on the surface. The surfaces with an almost complex structure or almost product structure are always integrable and have no local invariants. So studying the surfaces itself is not very interesting.

However the surfaces are embedded in the Grassmannian and it is very interesting to study the surfaces in the Grassmannian under the conformal isometry group of the Grassmannian. We call the analysis of the surfaces in the Grassmannian under the conformal isometry group of the Grassmannian the *microlocal analysis*. The reason for this is that we will see in the following chapters that first order systems and second order equations naturally define surfaces in the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$. For example a first order system is defined by a codimension 2 surface in $\mathrm{Gr}_2(TB)$ for a 4-dimensional base manifold B. In the fiber above $\mathrm{Gr}_2(T_bB)$ each point $b \in B$ the equation defines a surface. Moreover, the base transformations that leave a point b invariant act on $\mathrm{Gr}_2(T_bB)$ by conformal isometries. Here we write down the theory of surfaces to which the conformal quadratic form restricts to an indefinite quadratic form (the hyperbolic case). The elliptic case was already done by McKay in [51, Chapter 4]. By doing the hyperbolic case we can verify the results of McKay, and find differences between the elliptic and hyperbolic cases.

2.3.1 Hyperbolic surfaces in the Grassmannian

Let S be a surface in $\mathrm{Gr}_2(\mathbb{R}^4)$. At each point $s \in S$ the tangent space T_sS has dimension two and the conformal quadratic form restricts to a conformal quadratic form on T_sS. We call the point s *elliptic* if the quadratic form is positive or negative definite and *hyperbolic* if the form is non-degenerate indefinite. A surface for which all points are elliptic or hyperbolic is called an *elliptic surface* or *hyperbolic surface*, respectively.

Remark 2.3.1. In the literature the name hyperbolic surface is used for surfaces of constant negative curvature. The hyperbolic surfaces we introduce only have a conformal quadratic structure, so we cannot speak of the curvature of the surface. Another name for our surfaces would be conformal Lorentz surface. An introduction to Lorentz surfaces is given in Weinstein [72]. ⊘

For a hyperbolic surface the conformal quadratic form restricts on the tangent space of the surface to a non-degenerate conformal quadratic form of signature $(1, 1)$. The kernel of the quadratic form is given by two lines in the tangent space. The vectors in the two lines are called the *characteristic vectors*. Since these characteristic vectors depend smoothly on the point of the surface, the characteristic vectors locally define a pair of transversal rank one distributions. The integral curves of these distributions are called the *characteristic curves*.

Local coordinates

Let L_0 be the plane in $\mathrm{Gr}_2(\mathbb{R}^4)$ spanned by the two vectors $(1, 0, 0, 0)^T$ and $(0, 1, 0, 0)^T$. The elements in Σ_{L_0} (except for L_0 itself) are all of the form

$$\mathbb{R}\begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix} + \mathbb{R}\begin{pmatrix}-\rho\sin\alpha\\ \rho\cos\alpha\\ \cos\beta\\ \sin\beta\end{pmatrix},$$

with α, β, ρ arbitrary. We choose $M = \mathbb{R}(0, 0, 1, 0)^T + \mathbb{R}(0, 0, 0, 1)^T$ as a 2-plane that is transversal to L_0. We identify the open subset $\mathrm{Gr}_2^0(V, M)$ with the space of all 2×2-matrices using the correspondence

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \mapsto \mathbb{R}\begin{pmatrix}1\\ 0\\ a\\ c\end{pmatrix} + \mathbb{R}\begin{pmatrix}0\\ 1\\ b\\ d\end{pmatrix}.$$

With this correspondence we have

$$L_0 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \quad \Sigma_{L_0} \cap \mathrm{Gr}_2^0(V, M) = \{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \mid ad - bc = 0\}.$$

The tangent space at a point L in the Grassmannian is given in these local coordinates by the space of 2×2-matrices as well. The conformal quadratic form ξ on $T\,\mathrm{Gr}_2(\mathbb{R}^4)$ is given in these local coordinates at a point L by the determinant

$$\xi_L : T_L\,\mathrm{Gr}_2(\mathbb{R}^4) \to \mathbb{R} : \begin{pmatrix}A & B\\ C & D\end{pmatrix} \mapsto AD - BC.$$

If we describe a hyperbolic surface in local coordinates we will often use the coordinates introduced above. The general hyperbolic surface can be written in parametric form as

$$\begin{pmatrix}p(a,b) & q(a,b)\\ r(a,b) & s(a,b)\end{pmatrix}. \tag{2.7}$$

Since the group $\mathrm{GL}(V)$ acts transitively on the hyperbolic 2-planes in the tangent space, we can always find a local parameterization of the form

$$\begin{pmatrix} a & q(a,b) \\ r(a,b) & b \end{pmatrix}.$$

Another natural choice of local coordinates is a reparameterization of a, b such that the characteristic curves are given by the lines $a =$ constant and $b =$ constant.

Theorem 2.3.2. *Any hyperbolic surface in* $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^4) \cong S^+ \times S^-$ *can locally be written as the image of* $\phi : \mathbb{R} \times \mathbb{R} \to S^+ \times S^- : (x, y) \mapsto (s, t)$ *such that* $\partial s/\partial x \neq 0$ *and* $\partial t/\partial y \neq 0$.

Proof. This follows from the description of the isomorphism between in $\mathrm{Gr}_2(V)$ and $S^+ \times S^-$ in Section 2.1.2 and an analysis of the condition of hyperbolicity similar to the analysis in McKay [51, p. 20]. □

Hyperbolic tori

Every hyperbolic structure on a 4-dimensional vector space determines a hyperbolic surface. Let V be a vector space and K a hyperbolic structure on V. Then we define the *hyperbolic torus* as the set of all 2-planes that are K-invariant and satisfy the non-degeneracy condition that K restricted to the 2-plane is not equal to $\pm I$. We write $\mathrm{Gr}_2(V, K)$ for the hyperbolic torus associated to K. The elements of $\mathrm{Gr}_2(V, K)$ are called *hyperbolic lines*. This definition can be compared to the definition of the *complex lines* for a complex structure in McKay [51, p. 14]. The space of 2-dimensional complex-linear subspaces for a complex structure J on V will be written as $\mathrm{Gr}_2(V, J)$. The hyperbolic torus defined by a hyperbolic structure is topologically indeed a torus. If we let $V_\pm \subset V$ be the eigenspaces of the hyperbolic structure K, then $\mathrm{Gr}_1(V_+) \times \mathrm{Gr}_1(V_-) \to \mathrm{Gr}_2(V) : (l_1, l_2) \mapsto l_1 + l_2$ is an isomorphism $\mathrm{Gr}_1(V_+) \times \mathrm{Gr}_1(V_-) \to \mathrm{Gr}_2(V, K)$.

Example 2.3.3 (continuation of Example 2.2.6). The hyperbolic lines for the standard hyperbolic structure are given by a torus $\mathrm{Gr}_2(\mathbb{R}^4, K)$ in the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$. The hyperbolic lines can be parameterized using ϕ, θ as

$$\mathbb{R}\begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \\ 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 \\ \cos\phi \\ 0 \\ \sin\phi \end{pmatrix}.$$

Such a plane can also be represented by the bi-vector

$$\begin{aligned} q &= (\cos(\theta)e_1 + \sin(\theta)e_3) \wedge (\cos(\phi)e_2 + \sin(\phi)e_4) \\ &= \cos(\theta)\cos(\phi)e_1 \wedge e_2 + \cos(\theta)\sin(\phi)e_1 \wedge e_4 \\ &\qquad - \sin(\theta)\cos(\phi)e_2 \wedge e_3 + \sin(\theta)\sin(\phi)e_3 \wedge e_4 \in \Lambda^2(\mathbb{R}^4). \end{aligned}$$

The Plücker coordinates from Section 2.1.2 are equal to

$$\begin{aligned}
X_1 &= \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) = \cos(\theta - \phi), \\
Y_1 &= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) = \cos(\theta + \phi), \\
X_2 &= 0, \quad Y_2 = 0, \\
X_3 &= \cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) = \sin(\phi - \theta), \\
Y_3 &= \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi) = \sin(\theta + \phi).
\end{aligned} \qquad \oslash$$

Intersection curves

In this section we will analyze the intersection of a hyperbolic surface with the set Σ_{L_0} for L_0 a point on the hyperbolic surface.

Let S be a hyperbolic surface and L_0 a point on S. The manifold Σ_{L_0} has dimension three and has a singularity at L_0. We want to prove that locally the intersection of S and Σ_{L_0} looks like two lines intersecting transversally at L_0. First we introduce local coordinates around the point L_0 in the Grassmannian. The surface S is then given as a two-dimensional surface in the space of 2×2-matrices and the point L_0 corresponds to the zero matrix. Since the general linear group acts transitively on the hyperbolic tangent planes, we can arrange by a coordinate transformation that the tangent space to S is spanned at the point L_0 by the two tangent vectors

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In these coordinates we can parameterize the surface S using two coordinates a, b as

$$\sigma : U \subset \mathbb{R}^2 \to S : (a, b) \mapsto \begin{pmatrix} a & \phi(a, b) \\ \psi(a, b) & b \end{pmatrix},$$

with ϕ and ψ functions that vanish up to first order in a, b.

The manifold Σ_{L_0} is given by the 2-planes that have non-trivial intersection with L_0. These planes are precisely the planes for which the 2×2-matrix in local coordinates has zero determinant. Then $S \cap \Sigma_{L_0}$ is given by the condition $ab - \phi(a, b)\psi(a, b) = 0$. But the product $\phi(a, b)\psi(a, b)$ is of order 4 in a and b, hence by the Morse lemma (Lemma A.4.3) this set looks locally like the zero set of ab which is a cross at the origin. We call the two curves the *intersection curves* of the surface S through the point L_0.

Example 2.3.4. Let K be the standard hyperbolic structure on $\mathbb{R}^4$, see Example 2.2.6. Let $S = \mathrm{Gr}_2(\mathbb{R}^4, K)$ be the surface of hyperbolic lines in $\mathbb{R}^4$ for this hyperbolic structure. The elements of S can be represented by pairs of vectors

$$\begin{pmatrix} \cos\phi \\ 0 \\ \sin\phi \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \cos\psi \\ 0 \\ \sin\psi \end{pmatrix}.$$

In the local coordinates introduced above we have

$$S \cap \mathrm{Gr}_2^0(V, M) = \{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R} \}.$$

The intersection of S and Σ_{L_0} is easy to calculate and is given by

$$\begin{aligned} S \cap \Sigma_{L_0} \cap \mathrm{Gr}_2^0(V, M) &= \{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid ad = 0 \} \\ &= \{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \} \cup \{ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \mid d \in \mathbb{R} \}. \end{aligned}$$

So the intersection of Σ_{L_0} with S looks like a pair of lines intersecting at L_0.

The induced conformal quadratic form on the tangent space to S is given by AD. It is clear from this that the characteristic curves on S are spanned by the tangent vectors

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For every point L on the surface the characteristic curves through L are equal to the intersection curves through L defined by $\Sigma_L \cap S$. ⊘

We have proved that for a general hyperbolic surface S and point L_0 on this surface the intersection $\Sigma_{L_0} \cap S$ looks locally like two curves intersecting transversally at L_0. We can compare this pair of curves with the characteristic curves through the same point L_0 and this might provide us with some invariants. It can also happen that the characteristic curves through L_0 and the curves determined by $\Sigma_{L_0} \cap S$ are identical (see Example 2.3.4 above). In the local coordinates introduced before, the points L in Σ_{L_0} are determined by the condition $\det(L - L_0) = 0$. The tangent space at every point is also represented by the space of 2×2-matrices. The conformal quadratic form on the tangent space is also given by the determinant det. This shows that the characteristic vectors are those vectors in the tangent space for which $\det = 0$. It is then clear that the tangent vectors to the characteristic curves and the intersection curves $\Sigma_L \cap S$ are equal at every point L on the surface.

What happens outside the point L? The set Σ_L is still determined by the condition $\det L = 0$. The two examples below make clear that the intersection curves and characteristic curves through a point do not necessarily agree. At the point itself the curves agree up to first order, but outside the point they might diverge.

Example 2.3.5. Take the same local coordinates as in the previous example, but take S given by matrices of the form

$$\begin{pmatrix} a & 0 \\ \phi(a, b) & b \end{pmatrix}.$$

The characteristic curves are given by the lines $a = \text{constant}$ and $b = \text{constant}$. For every point (a, b) the union of the intersection curves through this point is equal to the set of points

$(\tilde{a}, \tilde{b})$ that satisfy

$$\det\left(\begin{pmatrix} \tilde{a} & 0 \\ \phi(\tilde{a}, \tilde{b}) & \tilde{b} \end{pmatrix} - \begin{pmatrix} a & 0 \\ \phi(a, b) & b \end{pmatrix}\right) = (\tilde{a} - a)(\tilde{b} - b) = 0.$$

The points that satisfy this condition are precisely given by the union of the two lines $\tilde{a} = a$ and $\tilde{b} = b$. So for this example the characteristic curves through the point (a, b) and the intersection curves through (a, b) are identical. $\oslash$

Example 2.3.6. We consider the surface defined in local coordinates for the Grassmannian by the matrices

$$\begin{pmatrix} a & a^2 \\ a^2 & b \end{pmatrix}.$$

The intersection curves through the origin follow from the equation

$$\det\begin{pmatrix} a & a^2 \\ a^2 & b \end{pmatrix} = ab - a^4 = a(b - a^3) = 0.$$

Here we can explicitly factorize the equation and this gives the intersection curves $a = 0$ and $b = a^3$. The characteristic vectors at a point a, b are

$$\begin{pmatrix} 1 & 2a \\ 2a & 4a^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can integrate these two vector fields to give the characteristic curves. Integrating the first vector field yields the characteristic curves $(a(t), b(t)) = (a_0 + t, b_0 + (4/3)((a_0 + t)^3 - (a_0)^3))$. Integration of the second vector field gives $(a(t), b(t)) = (a_0, b_0 + t)$. We immediately see that the intersection curves $a =$ constant overlap with the characteristic curves, but the other intersection curves do not overlap with the characteristic curves. $\oslash$

Compact hyperbolic surfaces

The condition that a surface in the Grassmannian is hyperbolic implies that at each point of the surface there are two characteristic lines in the tangent space. Locally, we can always choose a basis of the tangent space consisting of two non-zero vector fields tangent to these characteristic lines. We can locally make the choice of such a basis unique by choosing a metric on the surface, an order for the two characteristic lines (so we label one of the characteristics as the first and the other as the second characteristic line) and a positive direction for each of the characteristic lines

We can also choose a global metric for the surface (for example the metric induced from the diffeomorphism with $S^+ \times S^-$), but it is not always possible to make a global choice of order of the characteristics and directions. We can always pass to a cover of the surface on which the basis of vector fields is globally defined. We need at most a $8 : 1$ cover for this. First a $2 : 1$ cover for the ordering of the characteristic lined and then two times a $2 : 1$ cover for the direction of each of the characteristic lines.

Next consider the case of a connected compact hyperbolic surface. The covering surface is also compact and it is orientable. The covering surface has a trivial tangent space and this implies the surface has Euler characteristic zero; topologically the surface is a torus. The zero Euler characteristic is invariant under the cover map and hence the original surface is a compact surface with Euler characteristic zero and must be either a *torus* or a *Klein bottle*. The original surface is a torus if it is orientable and a Klein bottle if the surface is non-orientable. There exist explicit examples of compact surfaces in both the oriented and unoriented Grassmannian that are diffeomorphic to a Klein bottle, see the examples below. The hyperbolic torus defined by a hyperbolic structure is a compact hyperbolic surface that is homeomorphic to a torus.

Gluck and Warner [39] proved that every connected compact elliptic surface in the oriented Grassmannian can be deformed through elliptic surfaces to a Riemann sphere given by the complex lines for a complex structure on V. Also see the remarks in McKay [51, pp. 17, 23] on the relation between compact elliptic surfaces and fibrations of $S^3 \subset \mathbb{R}^4$ by great circles. For the hyperbolic surfaces the situation is not that simple. A compact hyperbolic surface can be topologically a torus or Klein bottle and these different types of surfaces can never be deformed into each other.

But even two compact hyperbolic surfaces that are both topologically a torus cannot always be deformed into each other. The oriented and unoriented Grassmannian are both connected but the oriented Grassmannian (which is the product of two spheres) is simply connected and hence the first fundamental group is trivial. The unoriented Grassmannian has fundamental group $\pi_1(\mathrm{Gr}_2(V)) \simeq \mathbb{Z}/2\mathbb{Z}$. There exist compact hyperbolic surfaces in the unoriented Grassmannian for which one of the generators of the fundamental group of the torus defines a non-trivial element in the fundamental group of the unoriented Grassmannian, but also compact hyperbolic surfaces for which the generators are all trivial in the fundamental group of the unoriented Grassmannian. Examples of both types are given in Example 2.3.10. The different types can not be deformed into each other.

If there are no topological obstructions against deformations (for example for the surfaces in the oriented Grassmannian), then it is not known whether it is possible to deform two surfaces into each other or not.

Example 2.3.7. We consider the oriented Grassmannian $\widetilde{\mathrm{Gr}}_2$ as the product of two spheres $S^+ \times S^-$. A family of immersed surfaces in the Grassmannian is given by

$$\Phi : (s, t) \mapsto \begin{pmatrix} \cos(s) \\ 0 \\ \sin(s) \end{pmatrix} \times \begin{pmatrix} \cos(t) \\ \sin(\alpha s)\sin(t) \\ \cos(\alpha s)\sin(t) \end{pmatrix}.$$

The tangent space at a point of the surface is spanned by the two vectors

$$\Phi_s = \frac{\partial \Phi}{\partial s} = \begin{pmatrix} -\sin(s) \\ 0 \\ \cos(s) \end{pmatrix} \times \begin{pmatrix} 0 \\ (\alpha)\cos(\alpha s)\sin(t) \\ -\alpha\sin(\alpha s)\sin(t) \end{pmatrix},$$

$$\Phi_t = \frac{\partial \Phi}{\partial t} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -\sin(t) \\ \sin(\alpha s)\cos(t) \\ \cos(\alpha s)\cos(t) \end{pmatrix}.$$

Solving the characteristic equation $\xi(a\Phi_s + b\Phi_t) = 0$, where ξ is the conformal quadratic from on the tangent space of the Grassmannian, yields

$$a = \pm \frac{b}{\sqrt{1 - \alpha^2 + \alpha^2\cos^2(s)}}.$$

For $|\alpha| < 1$ the surface has two distinct real characteristics at each point and hence the surface is hyperbolic.

For $\alpha = 0$ we have an embedded torus. The standard torus $T = \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}/(2\pi\mathbb{Z})$ is embedded as two great circles; the explicit parameterization is given by

$$T \to S^+ \times S^- : (s,t) \mapsto \left(\begin{pmatrix} \cos(s) \\ 0 \\ \sin(s) \end{pmatrix}, \begin{pmatrix} \cos(t) \\ 0 \\ \sin(t) \end{pmatrix} \right).$$

For $\alpha = 1/2$ the surface is a globally defined and compact surface K; topologically the surface is a Klein bottle. A $2:1$ cover of the torus $\tilde{T} = \mathbb{R}/(4\pi\mathbb{Z}) \times \mathbb{R}/(2\pi\mathbb{Z})$ to the Klein bottle $K \subset S^+ \times S^-$ is

$$\tilde{T} \to S^+ \times S^- : (s,t) \mapsto \left(\begin{pmatrix} \cos(s) \\ 0 \\ \sin(s) \end{pmatrix}, \begin{pmatrix} \cos(t) \\ \sin(s/2)\sin(t) \\ \cos(s/2)\sin(t) \end{pmatrix} \right). \qquad \oslash$$

Example 2.3.8. We consider the oriented Grassmannian $\widetilde{\mathrm{Gr}}_2(V)$ as the product $S^+ \times S^-$ of two spheres. The unoriented Grassmannian $\mathrm{Gr}_2(V)$ is isomorphic to the quotient of $S^+ \times S^-$ by the involution $(x,y) \mapsto (-x,-y)$. Let T be the torus $\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}/(2\pi\mathbb{Z})$ and define the immersion

$$\Phi : T \to S^+ \times S^- : (s,t) \mapsto \left(\begin{pmatrix} \sqrt{1-z^2}\cos(2s) \\ \sqrt{1-z^2}\sin(2s) \\ z \end{pmatrix}, \begin{pmatrix} \cos(t) \\ \cos(s)\sin(t) \\ \sin(s)\sin(t) \end{pmatrix} \right), \qquad \oslash$$

with $0 < z < (1/2)\sqrt{3}$ a constant. The image of Φ is a Klein bottle in the oriented Grassmannian. The map Φ is a 2-fold over of this Klein bottle. The projection to the unoriented Grassmannian gives an embedding of the Klein bottle in the unoriented Grassmannian.

Example 2.3.9. Let $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to S^+$ be an embedding of the circle into the 2-sphere with the properties $\gamma(s+\pi) = -\gamma(s)$ for all s and $|\gamma'(s)| > 1$ for all s. Such embeddings are easy to construct by taking deformations of great circles and then reparameterizing by arc length. We define $T = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ and $\Phi : T \to \widetilde{\mathrm{Gr}}_2(V)$ by

$$\Phi(s,t) = \left(\gamma(s), (\cos s \cos t, \sin s \cos t, \sin t)^T\right).$$

Then

$$a\frac{\partial\Phi}{\partial s} + b\frac{\partial\Phi}{\partial t} = \left(a\gamma'(s), \begin{pmatrix} -a \sin s \cos t - b \cos s \sin t \\ a \cos s \cos t - b \sin s \sin t \\ b \cos t \end{pmatrix}\right).$$

The square of the length of the first vector minus the square of the length of the second vector equals $a^2|\gamma'(s)|^2 - (a^2\cos^2(t) + b^2) = a^2(|\gamma'(s)|^2 - \cos^2(t)) - b^2$. This is an indefinite non-degenerate quadratic form in a, b at all points. Hence the surface defined by Φ is a hyperbolic surface. The image of the torus T is a torus in the oriented Grassmannian. The projection of $\widetilde{\mathrm{Gr}}_2(V)$ to $\mathrm{Gr}_2(V)$ induces a $2:1$ cover of the torus over a Klein bottle in the unoriented Grassmannian. $\oslash$

Example 2.3.10 (Compact hyperbolic surfaces and the fundamental group). Let T be the torus $\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}/(2\pi\mathbb{Z})$ and let z be a constant with $0 < z < 1/2$. We define two compact hyperbolic surfaces in $\mathrm{Gr}_2(V) = (S^+ \times S^-)/(-I, -I)$ by

$$\Phi^1 : T \to \mathrm{Gr}_2(V) : (s,t) \mapsto \left(\begin{pmatrix} \sqrt{1-z^2}\cos(s) \\ \sqrt{1-z^2}\sin(s) \\ z \end{pmatrix}, \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}\right),$$

$$\Phi^2 : T \to \mathrm{Gr}_2(V) : (s,t) \mapsto \left(\begin{pmatrix} \cos(s/2) \\ \sin(s/2) \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(t+s/2) \\ \sin(t+s/2) \\ 0 \end{pmatrix}\right).$$

Both maps Φ^1, Φ^2 are embeddings of the torus T into the unoriented Grassmannian.

Let γ be the curve in T defined by $s \mapsto (s, 0)$. Then γ defines a non-trivial element $[\gamma]$ in the fundamental group of T. The embedding Φ^j induces a homomorphism Φ^j_* from the fundamental group $\pi_1(T)$ to $\pi_1(\mathrm{Gr}_2(V))$. The image $(\Phi^1)_*([\gamma])$ is trivial in $\pi_1(\mathrm{Gr}_2(V))$, the image $(\Phi^2)_*([\gamma])$ is non-trivial in $\pi_1(\mathrm{Gr}_2(V))$. $\oslash$

2.3.2 Geometrically flat surfaces

We define a hyperbolic surface to be *geometrically flat* if the characteristic curves and the intersection curves are identical. From Example 2.3.4 it is clear the the hyperbolic tori are geometrically flat. The converse is not true. For example the surfaces parameterized in Example 2.3.5 are all geometrically flat. But these surfaces depend on the arbitrary function $\phi(a,b)$. The hyperbolic structures are parameterized by 8 coordinates, they are isomorphic to $\mathrm{GL}(4,\mathbb{R})/\mathrm{GL}(2,\mathbb{D})$. The condition that a point is in the hyperbolic torus for a hyperbolic structure imposes two restrictions. The hyperbolic tori through a given point of the Grassmannian depend on 6 coordinates. So not all these surfaces can be hyperbolic tori and this proves the class of all geometrically flat surfaces is much larger then the class of all hyperbolic tori.

To analyze the structure of geometrically flat surfaces we start with en elementary lemma.

Lemma 2.3.11. *Let $V = \mathbb{R}^4$ and let L_1, L_2, L_3 be 2-dimensional linear subspaces such that* $\dim L_1 \cap L_2 = \dim L_1 \cap L_3 = \dim L_2 \cap L_3 = 1$. *Then the L_j are all contained in a 3-dimensional linear subspace $L = L_1 + L_2 + L_3$ or the three subspaces have a 1-dimensional linear subspace $l = L_1 \cap L_2 \cap L_3$ in common. It is also possible that L_1, L_2, L_3 are contained in a* 3*-dimensional subspace and have a 1-dimensional line in common.*

Proof. Assume that $L_1 \cap L_2 \cap L_3 = \{0\}$, so the subspaces have no line in common. Pick vectors e_1, e_2, e_3 in V such that $L_1 \cap L_2 = \mathbb{R}e_1$, $L_1 \cap L_3 = \mathbb{R}e_2$ and $L_2 \cap L_3 = \mathbb{R}e_3$. We cannot have $\mathbb{R}e_1 = \mathbb{R}e_2$ since this would imply that $\mathbb{R}e_1 \subset L_1 \cap L_2 \cap L_3$. Hence $L_1 = \mathbb{R}e_1 + \mathbb{R}e_2$. Since $\{0\} = L_1 \cap L_2 \cap L_3 = L_1 \cap \mathbb{R}e_3$ we see that e_3 is not in the span of e_1, e_2. Hence the vectors e_1, e_2, e_3 are linearly independent. From the construction of e_1, e_2, e_3 it is clear that $L_1 + L_2 + L_3 = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$ and that $\dim(L_1 + L_2 + L_3) = 3$.□

Let S be a geometrically flat surface in $\mathrm{Gr}_2(V)$. Let L_1, L_2, L_3 be three different points on the same characteristic curve γ. Since the surface is geometrically flat, one of the intersection curves through the point L_1 must be identical to the characteristic curve γ. Therefore both L_2 and L_3 must have non-zero intersection with L_1 and for the same reason L_2 and L_3 must have non-zero intersection. Recall that the points L_k are elements of the Grassmannian and hence 2-dimensional linear subspaces of V. Because the points L_1, L_2 and L_3 are different points, the intersections must be 1-dimensional and we can apply Lemma 2.3.11. This leads to the conclusion that there are three types of characteristic curves γ on a geometrically flat surface.

1) All points L on γ have a line l_1 in common and are contained in a three-dimensional subspace l_3.

2) All points L on γ have a line l_1 in common. The points L are not contained in a subspace of dimension three.

3) All points L on γ are contained in a three-dimensional subspace l_3. The points on γ do not have a line in common.

We say a characteristic curve is of type ($2'$) if the characteristic curve is either of type (1) or of type (2). We say a characteristic curve is of type ($3'$) if the characteristic curve is either of type (1) or of type (3). For a hyperbolic surface the type of the characteristic curves does not need to be constant. An example of such a surface is given in Example 2.3.12.

Let $S : (a, b) \mapsto x(a, b) \in \mathrm{Gr}_2(V)$ be a hyperbolic surface such that the characteristic curves are given by the equations $a =$ constant and $b =$ constant. Whenever we have a hyperbolic surface parameterized in this way, we will call the curves defined by $b =$ constant the *horizontal characteristic curves* and the curves $a =$ constant the *vertical characteristic* *curves*. For a surface with (locally) constant type there are nine possibilities: the horizontal characteristic curves can have type (1), (2) or (3) and the vertical characteristic curves as well. If we allow to switch the characteristic curves, then there are only six types. We will say a geometrically flat surface if of type (i, j) if the horizontal characteristic lines are of type (i) and the vertical characteristics of type (j).

Example 2.3.12 (Changing type). Let $P_{x_0,x_1}(x)$ be a smooth bump function that is zero outside the region $x_0 < x < x_1$ and non-zero inside this region. For example we can take

$$P_{x_0,x_1}(x) = \begin{cases} 0 & \text{for } x \leq x_0, \\ \exp\left(\frac{-1}{(x-x_0)^2}\right)\exp\left(\frac{-1}{(x_1-x)^2}\right) & \text{for } x_0 < x < x_1, \\ 0 & \text{for } x_1 \leq x. \end{cases}$$

We then define $\phi_1(a) = P_{0,1}(a)$, $\phi_2(a) = P_{2,3}(a)$, $\psi_1(b) = P_{0,1}(b)$, $\psi_2(b) = P_{2,3}(b)$. Let S be the surface be defined in local coordinates (see Section 2.3.1) by the matrix

$$A = \begin{pmatrix} a & \phi_1(a)\psi_1(b) \\ \phi_2(a)\psi_2(b) & b \end{pmatrix}.$$

The embedding $S \to \mathrm{Gr}_2(V)$ defined by the matrix A in local coordinates is denoted by $x : (a, b) \mapsto x(a, b)$. The embedding x defines a hyperbolic surface and at each point (a, b) the matrices $\partial A/\partial a$ and $\partial A/\partial b$ are singular. This means that the characteristic curves are given by the lines $a =$ constant and $b =$ constant. To show that the intersection curves coincide with the characteristic curves consider an arbitrary point (a, b). The point $x(\tilde{a}, \tilde{b})$ is contained in $\Sigma_{x(a,b)}$ if and only if $\det(A - \tilde{A}) = 0$. Consider the points $(\tilde{a}, b)$. For these points we have

$$\begin{aligned} \det(A(a, b) - A(\tilde{a}, b)) &= \det \begin{pmatrix} a - \tilde{a} & (\phi_1(a) - \phi_1(\tilde{a}))\psi_1(b) \\ (\phi_2(a) - \phi_2(\tilde{a}))\psi_2(b) & 0 \end{pmatrix} \\ &= (\phi_1(a) - \phi_1(\tilde{a}))\,(\phi_2(a) - \phi_2(\tilde{a}))\,\psi_1(b)\psi_2(b). \end{aligned}$$

Since $\psi_1(b)\psi_2(b)$ is identically zero this shows that all points $(\tilde{a}, b)$ are in $\Sigma_{x(a,b)}$. This proves that the characteristic curves $b =$ constant coincide with the intersection curves. A similar analysis shows that also the lines $a =$ constant coincide the with intersection curves.

The hyperbolic surface in this example has changing type of characteristics. We denote the different types by (i, j). The pair (i, j) means that the horizontal characteristics ($b =$ constant) have type (i) and the vertical characteristics ($b =$ constant) have type (j). In Figure 2.1 the different regions on the surface are separated by black lines and the types are indicated.

For example in the region $1 \leq a \leq 2$, $1 \leq b \leq 2$ the surface has type (2, 3). In this region the surface is parameterized by

$$A = \begin{pmatrix} a & 0 \\ \phi_2(a)\psi_2(b) & b \end{pmatrix}.$$

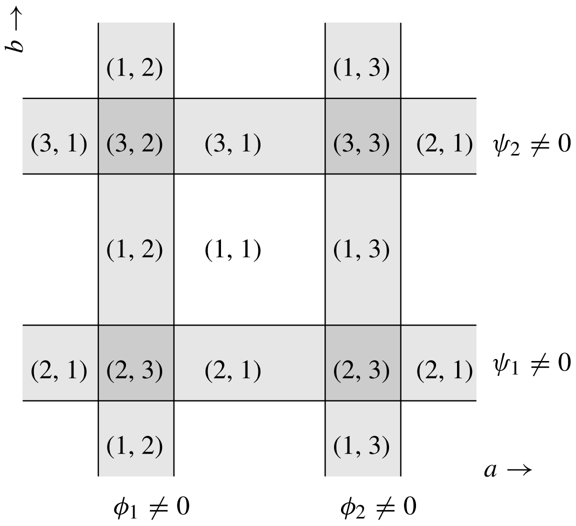

Figure 2.1: Geometrically flat surface with changing type of curves

The horizontal characteristics have b constant. The points on a line b = constant are 2-planes that all have the line spanned by the vector $(0, 1, 0, b)^T$ in common. The vertical characteristics have a constant. The points on such a characteristic are all 2-planes in the 3-dimensional subspace spanned by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ a \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This single example shows that all possible combinations of type (i, j) exist for hyperbolic surfaces. $\oslash$

Example 2.3.13 (Geometrically flat surface of type (2′, 2′)). Let γ and δ be two curves in $\mathrm{Gr}_1(\mathbb{R}^4)$ and define $\Gamma(s, t) = \gamma(s) + \delta(t)$. Assume that $\gamma(0) \neq \delta(0)$ and the tangent map of Γ at $(0, 0)$ is injective. Then Γ (locally near $L_0 = \Gamma(0, 0)$) defines a surface S in $\mathrm{Gr}_2(\mathbb{R}^4)$. If the tangent plane $T_{L_0}S$ to the surface at L_0 is a hyperbolic tangent plane, then S is a hyperbolic surface near L_0.

This surface has the property that every point $L = \Gamma(s, t_0)$ on the curve $\phi_{t_0} : s \mapsto \Gamma(s, t_0)$ contains the line $\delta(t_0)$. Hence the intersection curves through the points L on this curve are all tangent to the curve ϕ_{t_0}. Since the intersection curves are always tangent to the characteristic curves this proofs that ϕ_{t_0} is a characteristic curve for the surface. In a similar way it follows that the curves $\psi_{s_0} : t \mapsto \Gamma(s_0, t)$ are characteristic curves and intersection curves for the points on ψ_{s_0}.

This surface is geometrically flat and the type is $(2',2')$ because the points on the characteristic line ϕ_{t_0} have the 1-dimensional linear subspace $\delta(t_0)$ in common and the points on the characteristic line ψ_{s_0} have the 1-dimensional linear subspace $\gamma(s_0)$ in common. ⊘

Example 2.3.14 (Geometrically flat surfaces in local coordinates). Choose local coordinates a, b such that the characteristic curves are given by $a =$ constant and $b =$ constant. Parameterize the surface in local coordinates as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ p(a,b) & q(a,b) \\ r(a,b) & s(a,b) \end{pmatrix}.$$

The condition that the surface is geometrically flat, is precisely that for all points (a, b) and $(\tilde{a}, \tilde{b})$ for which either $a = \tilde{a}$ or $b = \tilde{b}$ we have

$$(\tilde{p} - p)(\tilde{s} - s) - (\tilde{r} - r)(\tilde{q} - q) = 0.$$

One can check that the examples in local coordinates of geometrically flat surfaces in this section all satisfy this condition. ⊘

Type $(2', 3')$. Let S be a geometrically flat hyperbolic surface in $\mathrm{Gr}_2(V)$ of type $(2', 3')$. We assume that we have introduced coordinates a, b such that the characteristic curves of type $(2')$ are given by $b =$ constant (the horizontal curves) and the characteristic curves of type $(3')$ are given by $a =$ constant (the vertical curves).

For every point $x(a, b) \in \mathrm{Gr}_2(V)$ the points on the horizontal characteristic curve γ_1 through $x(a, b)$ have a line $l_1(b)$ in common. The points on the vertical characteristic curve γ_2 are all contained in a three-dimensional subspace $l_3(a)$. The lines $l_1(b)$ and the three-dimensional spaces $l_3(a)$ satisfy the relation

$$l_1(b) \subset x(a, b) \subset l_3(a).$$

This relation implies that

$$\bigcup_b l_1(b) \subset \bigcap_a l_3(a).$$

We use the notation $\sum_b l_1(b)$ to denote the span of the elements in $\bigcup_b l_1(b)$. Then it is clear that $\sum_b l_1(b)$ is a linear subspace of $\bigcap_a l_3(a)$.

The lines $l_1(b)$ and the three-dimensional subspaces $l_3(a)$ must both vary as we vary a and b. For example if $l_1(b)$ is constant near $x_0 = (a_0, b_0)$, then near x_0 all points on the surface have a single line $l_1 = l_1(b_0)$ in common. But then near x_0 the intersection $\Sigma_{x(a_0,b_0)} \cap S$ is equal to S and this is not possible. This implies that there is a unique two-dimensional linear subspace L such that

$$\sum_b l_1(b) = L = \bigcap_a l_3(a). \tag{2.8}$$

The special point L is not a point on the surface. If $L = \mathbb{R}l_1(b) + \mathbb{R}l_1(\tilde{b})$ then this would imply that the two intersection curves corresponding to $l_1(b)$ and $l_1(\tilde{b})$ both intersect L. But the surface is of type (2′,3′) so no point has two characteristic curves of type (2′).

Example 2.3.15 (Compact surfaces of type (2′, 3′)). In this example we will make a construction of a large class of compact hyperbolic surfaces of class (2′, 3′). Recall that for any surface of type (2′, 3′) there is a unique 2-plane L that satisfies the equation (2.8). We define

$$F_L = \{ (l_1, l_2, l_3) \in \mathrm{Gr}_1(V) \times \mathrm{Gr}_2(V) \times \mathrm{Gr}_3(V) \mid l_1 \subset L,\ l_1 \subset l_2 \subset l_3,\ L \subset l_3 \}. \tag{2.9}$$

The first component l_1 will be related to the common line of points on the characteristics of type (2′) and the third component l_3 will be related the the three-dimensional subspace spanned by the points on the characteristics of type (3′). The space F_L is a smooth manifold of dimension 3.

We will analyze the two projections

$$\pi_2 : F_L \to \mathrm{Gr}_2(V) : (l_1, l_2, l_3) \mapsto l_2, \tag{2.10}$$
$$\pi_{1,3} : F_L \to \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L) : (l_1, l_2, l_3) \mapsto (l_1, l_3/L). \tag{2.11}$$

The projection $\pi_{1,3} : F_L \to \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$ is surjective. The fiber above a point $(l_1, l_3/L)$ is diffeomorphic to $\mathrm{Gr}_1(l_3/l_1)$. This shows $\pi_{1,3}$ is a $\mathbb{P}^1$ bundle over $\mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$.

At the points $l_2 \neq L$ in the image of π_2 we have

$$\pi_2^{-1}(l_2) = \{ (l_1, l_2, l_3) \in F_L \mid l_1 = l_2 \cap L,\ l_3 = l_2 + L \}.$$

So π_2 is injective over the complement of L in $\mathrm{Gr}_2(V)$. The rank of $T\pi_2$ over this complement is 3. For the special point L we have

$$\pi_2^{-1}(L) = \{ (l_1, L, l_3) \in F_L \mid l_1 \in \mathrm{Gr}_1(L),\ L \subset l_3 \in \mathrm{Gr}_1(V/L) \}.$$

This shows that $T\pi_2$ has rank one at the points in F_L that project to L. We have an isomorphism

$$\pi_2^{-1}(L) \to \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L) : (l_1, L, l_3) \mapsto (l_1, l_3/L).$$

This shows that the inverse image $\pi_2^{-1}(L)$ defines a special section of the bundle $\pi_{1,3} : F_L \mapsto \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$.

Let $F'_L = \{ (l_1, l_2, l_3) \in F_L \mid l_2 \neq L \}$ and let $\pi'_{1,3}$ be the restriction of $\pi_{1,3}$ to the bundle F'_L. The fiber above a point (l_1, l_3) is isomorphic to $\mathrm{Gr}_1(l_3/l_1) \setminus L \cong \mathbb{P}^1 \setminus \{0\}$. This gives $\pi'_{1,3} : F'_L \to \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$ the structure of an affine line bundle.

For any (local) section σ of the bundle $\pi'_{1,3}$ we can consider the composition $\pi'_2 \circ \sigma : \mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L) \to \mathrm{Gr}_2(V)$. The map is embedding since $\pi'_2 : F' \to \mathrm{Gr}_2(V)$ has rank 3 and is injective. Global sections of this bundle exist. Take for example a transversal 2-plane M such that $V = L \oplus M$. A global section of $\pi'_{1,3}$ is given by $(l_1, l_3) \mapsto (l_1, l_1 + M \cap l_3, l_3)$.

This is a map from a torus to the Grassmannian. The hyperbolic surface defined by the composition of this section with π_2' is the hyperbolic torus $\mathrm{Gr}_2(V, K)$ for the hyperbolic structure K defined by $V = L \oplus M$. After a choice of global section the line bundle F_L' becomes a rank one vector bundle over $\mathrm{Gr}_1(L) \times \mathrm{Gr}_1(V/L)$. The sections of this bundle can locally be parameterized by exactly one function of two variables. The global sections define compact geometrically flat surfaces of type $(2', 3')$. $\oslash$

Example 2.3.16 (continuation of Example 2.3.5). In local coordinates the construction in the previous example leads to a surface as described in Example 2.3.5. For a point (a, b) on the surface the line in common to all points on the characteristic $b =$ constant is given by $(0, 1, 0, b)^T$. The 3-dimensional spaces $l_3(a)$ given by $a =$ constant are given by the span of

$$\begin{pmatrix} 1 \\ 0 \\ a \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The unique 2-plane equal to the union of the $l_1(b)$ and equal to the intersection of the $l_3(a)$ is $L = \mathbb{R}e_1 + \mathbb{R}e_3$. The coordinate b is a local coordinate for $\mathrm{Gr}_1(L)$ and a is a local coordinate for $\mathrm{Gr}_1(V/L)$. $\oslash$

Type (1, 1). We will prove that any geometrically flat surface S in $\mathrm{Gr}_2(V)$ of type $(1, 1)$ is locally given by the hyperbolic lines for a unique hyperbolic structure on V. Note that every surface of this type is a special case of a surface of type $(2', 3')$. Locally we can make a choice of horizontal and vertical characteristic lines. The horizontal lines are both of type $(2')$ and of type $(3')$ and the same holds for the vertical characteristics. This means that by switching the characteristic lines there are two ways in which we can view a surface of type $(1, 1)$ as a $(2', 3')$ surface. For any point L on the surface let $l_1^+(L)$ be the unique line that all points on the horizontal characteristic curve have in common and $l_1^-(L)$ be the unique line that all points on the vertical characteristic curve have in common. In a similar way let $l_3^+(L)$ be the unique 3-dimensional space spanned by all points in the horizontal curve and $l_3^-(L)$ the unique 3-dimensional space spanned by all points in the vertical curve. Recall that the surfaces of type $(2', 3')$ all have a unique 2-plane L associated to them. Since a surface of type $(1, 1)$ can be viewed in two different ways as a $(2', 3')$ surface we find two invariant 2-planes L^+ and L^-. To be more precise: we define L^+ as the unique 2-plane such that $\sum l_1^+(L) = L^+ = \bigcap l_3^-(L)$ and L^- as the unique 2-plane such that $\sum l_1^-(L) = L^- = \bigcap l_3^+(L)$. It is not difficult to see that $L^+ \cap L^- = \{0\}$ and hence the two planes define a hyperbolic structure K on $V = L^+ \oplus L^-$.

Let M be a point in the surface S. Then $L^+ \cap M = l_1^+(M)$ and $L^- \cap M = l_1^-(M)$. Hence M is a point in the hyperbolic torus $\mathrm{Gr}_2(V, K)$. Since S and $\mathrm{Gr}_2(V, K)$ both have dimension two this proves that locally $S = \mathrm{Gr}_2(V, K)$.

2.3.3 Normal form calculations

In this section we calculate a normal form for the hyperbolic surfaces at a special point. The group acting on the surface is the group of conformal isometries of $\mathrm{Gr}_2(V)$. In the next section we will give a geometric interpretation of this normal form calculation. The geometric picture is needed to make the connection to first order systems and second order equations in Chapter 5 and Chapter 6.

Zero and first order. We want to bring a hyperbolic surface S in $\mathrm{Gr}_2(V)$ into a normal form using the group $\mathrm{GL}(V)$. We could also use the projective group $\mathbb{P}\,\mathrm{GL}(V)$ since the scalar multiplications do not act on $\mathrm{Gr}_2(V)$. In the calculations below it is always easy to translate the actions of $\mathrm{GL}(V)$ and $\mathbb{P}\,\mathrm{GL}(V)$ into each other. Since the group acts transitively on the points in $\mathrm{Gr}_2(V)$ and on the hyperbolic tangent spaces at that point (see Appendix A.5), we can always choose a basis e_1, e_2, e_3, e_4 for V such that the point $L \in S$ is given by $\mathbb{R}e_1 + \mathbb{R}e_2$ and the tangent space to the surface at L is given by the linear maps in $\mathrm{Lin}(L, V/L)$ that are diagonal matrices with respect to the bases e_1, e_2 for L and e_3, e_4 for V/L.

In the local coordinates introduced in Section 2.3.1 the surface is given by the matrices

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{2.12}$$

The matrix A corresponds to the 2-plane spanned by the two vectors

$$\begin{pmatrix} 1 \\ 0 \\ p \\ r \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ q \\ s \end{pmatrix}.$$

The matrix A can also be viewed as a linear map $L \to M$, with $M = \mathbb{R}e_3 + \mathbb{R}e_4$. In this view the matrix A corresponds to the 2-plane spanned by vectors of the form $x + Ax$, $x \in L$. The special point L corresponds to the zero matrix. If we use p, s as local coordinates, then q and r are functions of p and s. We will bring the surface in normal form by constructing a normal form for the Taylor expansions of $q(p, s)$ and $r(p, s)$. The normalization at order zero was the choice of special point L. This normalization corresponds to $q(0, 0) = 0$ and $r(0, 0) = 0$. The normalization at order one was the choice of tangent space to S at L. This corresponds to $q = \mathcal{O}(p, s)^2$, $r = \mathcal{O}(p, s)^2$.

The group $\mathrm{GL}(V)$ can be parameterized by the 4×4 matrices

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix},$$

with $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ all 2×2-matrices. The subgroup $H_0 = \mathrm{GL}(V)_L$ that leaves invariant L is given by the matrices with $\tilde{c} = 0$. We compute the action of $\mathrm{GL}(V)_L$ on the tangent space $T_L S$. Let

$$g = \begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{d} \end{pmatrix}.$$

Then g acts on A as

$$g \cdot A = \tilde{d}A(\tilde{a} + \tilde{b}A)^{-1}.$$

Note this action might not be well-defined for all g since we are working in local coordinates for $\mathrm{Gr}_2^0(V, M)$, but it is well-defined for elements g near the identity. On the tangent space this induces the action $B \mapsto \tilde{d}B\tilde{a}^{-1}$. This conformal action is transitive on the hyperbolic planes and we can always arrange that B is a diagonal matrix.

The structure group that leaves invariant L and $T_L S$ is the group $H_1 = \mathrm{GL}(V)_{L,T_L S}$ of matrices

$$\begin{pmatrix} a & \tilde{b} \\ 0 & d \end{pmatrix} \in \mathrm{GL}(V), \tag{2.13}$$

with either $a, d \in D$ or a, d both anti-diagonal, i.e., $a, b \in LD$. This group has dimension 8 (or dimension 7 if we are working with the projective group).

Second order. The space of second order contacts to a hyperbolic surface for which the first order part is in normal form has dimension 6. The action of the group (2.13) induces an action on this space by affine transformations. If we use the local coordinates (2.12), then the first order normalizations correspond to $q = q_{11}p^2/2 + q_{12}ps + q_{22}s^2/2 + \mathcal{O}(p, s)^3$, $r = r_{11}p^2/2 + r_{12}ps + r_{22}s^2/2 + \mathcal{O}(p, s)^3$. The action on A is given by

$$A \mapsto dA(a + \tilde{b}A)^{-1} = dAa^{-1} - dAa^{-1}\tilde{b}Aa^{-1} + \mathcal{O}(|A|^3).$$

We will calculate the action of the connected component of the group H_1. The action of the other component can be calculated in a similar fashion. We write

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Working out this action using p, s as coordinates and only keeping terms of order 2 and lower yields

$$\begin{aligned} A \mapsto \tilde{A} &= \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{r} & \tilde{s} \end{pmatrix} \\ &= \begin{pmatrix} d_1 a_1^{-1} p - d_1 a_1^{-2} p^2 b_{11} & d_1 a_2^{-1} q - d_1 a_2^{-1} a_1^{-1} b_{12} ps \\ d_2 a_1^{-1} r - d_2 a_1^{-1} a_2^{-1} b_{21} ps & d_2 a_2^{-1} s - d_2 a_2^{-2} b_{22} s^2 \end{pmatrix} + \mathcal{O}(p, s)^3. \end{aligned}$$

We use $\tilde{p} = d_1 a_1^{-1} p - d_1 a_1^{-2} b_{11} p^2$ and $\tilde{s} = d_2 a_2^{-1} s - d_2 a_2^{-2} b_{22} s^2$ as new local coordinates. Since $\tilde{p}, \tilde{s}$ are diagonal in p, s up to first order, this preserves the normal form. We can express $\tilde{q}$ and $\tilde{r}$ in the new coordinates $\tilde{p}, \tilde{s}$; the final result is

$$\begin{aligned} &\tilde{q}_{11} = (a_1)^2 a_2^{-1} d_1^{-1} q_{11}, \quad &&\tilde{q}_{12} = a_1 d_2^{-1} q_{12} - d_2^{-1} b_{12}, \quad &&\tilde{q}_{22} = d_1 d_2^{-2} a_2 q_{22}, \\ &\tilde{r}_{11} = d_2 d_1^{-2} a_1 r_{11}, &&\tilde{r}_{12} = a_2 d_1^{-1} r_{12} - d_1^{-1} b_{21}, &&\tilde{r}_{22} = a_1^{-1} (a_2)^2 d_2^{-1} r_{22}. \end{aligned} \tag{2.14}$$

The action is indeed by affine transformations. Note that the group coefficients b_{11}, b_{22} do not appear in these expressions so this part of the group does not act on the second order contact. Using the group parameters b_{12}, b_{21} we can always arrange that $\tilde{q}_{12} = \tilde{r}_{12} = 0$. On the remaining four coefficients the generic orbits have dimension three. There is one invariant given by

$$I = \frac{q_{11} r_{22}}{r_{11} q_{22}}. \tag{2.15}$$

The invariant is a rational function in the coefficients of the second order jets of a hyperbolic surface. If $r_{11}q_{22} = 0$ but $q_{11}r_{22} \neq 0$, then we say the invariant takes the value ∞. If both $q_{11}r_{22} = 0$ and $r_{11}q_{22} = 0$, then this invariant is not well-defined (by making small perturbations the invariant can have any possible value).

Remark 2.3.17. We will analyze the action of the other component of H_1 on the second order coefficients. Let

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in H_1.$$

The action on a surface in local coordinates is

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} s & r \\ q & p \end{pmatrix}.$$

If we write $\tilde{p} = s, \tilde{s} = p, \tilde{q} = r, \tilde{r} = q$ and assume that q, r are normalized and of the form

$$q = q_{11}p^2/2 + q_{12}ps + q_{22}s^2/2, \quad r = r_{11}p^2/2 + r_{12}ps + q_{22}r^2/2,$$

then

$$\begin{aligned} \tilde{q}_{11} &= r_{22}, & \tilde{q}_{12} &= r_{12}, & \tilde{q}_{22} &= r_{11}, \\ \tilde{r}_{11} &= q_{22}, & \tilde{r}_{12} &= q_{12}, & \tilde{r}_{22} &= q_{11}. \end{aligned}$$

The new invariant $\tilde{I}$ is equal to the original invariant

$$\tilde{I} = \frac{\tilde{q}_{11}\tilde{r}_{22}}{\tilde{r}_{11}\tilde{q}_{22}} = \frac{q_{11}r_{22}}{r_{11}q_{22}} = I.$$

So I is really invariant under the full group H_1. $\oslash$

Third and higher order. We will conclude the normal form calculations by showing that for generic structures (all terms $q_{11}, q_{22}, r_{11}, r_{22}$ unequal to zero, or equivalently the invariant I is well-defined, non-zero and finite) the group acts effectively. If we are at a generic point, then we can normalize the second order coefficients to $q_{12} = r_{12} = 0$, $q_{11} = r_{11} = q_{22} = 1$ and $r_{22} = I$. The structure group reduces to the group H_S consisting of matrices

$$g = \phi \begin{pmatrix} I & b \\ 0 & I \end{pmatrix} \in \mathrm{GL}(V),$$

with $\phi \in \mathbb{R}^*$ and $b \in D$. The scalar factor ϕ is not important since the group only acts by projective transformations.

The action on the third order part is relatively easy to calculate because the structure group has reduced to such a small group. The action of g on the matrix A is

$$g : A \mapsto \tilde{A} = A(I + bA)^{-1} = A - AbA + AbAbA + \mathcal{O}(|A|)^4. \tag{2.16}$$

At the special point L we have $A = 0$ and the first order part of A is diagonal. Therefore we can write $A = A_1 + A_2 + A_3$ with $A_1 \in D$ and the second and third order parts A_2 and A_3 anti-diagonal and homogeneous of degree 2 and 3 in A_1, respectively. We then expand the expression (2.16) and group the diagonal and anti-diagonal terms

$$\begin{aligned} \tilde{A} &= A - AbA + AbAbA + \mathcal{O}(|A|)^4 \\ &= (A_1 - bA_1^2 + b^2 A_1^3) + (A_2 + A_3 - (A_1 b A_2 + A_2 b A_1)) + \mathcal{O}(|A|)^4. \end{aligned}$$

The new diagonal part is given by $\tilde{A}_1 = A_1 - bA_1^2 + b^2 A_1^3$. The old part can be expressed in the new part as $A_1 = \tilde{A}_1 + b\tilde{A}_1^2 + b^2 \tilde{A}^3 + \mathcal{O}(|\tilde{A}|)^4$. The second order part is given by A_2. This term is homogeneous of degree 2 in A_1 and hence also homogeneous of degree 2 in $\tilde{A}_1$. So the action of H_S does not change the second order part, as is required. In particular we have $\tilde{q} = q_{11}\tilde{p}^2/2 + q_{22}\tilde{s}^2/2$, $\tilde{r} = r_{11}\tilde{p}^2/2 + r_{22}\tilde{s}^2/2$.

The third order part is formed by three terms: A_2 has a contribution, A_3 has a contribution and $A_1 b A_2 + A_2 b A_1$ has a contribution. Let us assume the first order part A_1 was diagonal of the form (p, s) and the second order part A_2 was of the form

$$\begin{pmatrix} 0 & q_{11}p^2/2 + q_{22}s^2/2 \\ r_{11}p^2/2 + r_{22}s^2/2 & 0 \end{pmatrix}.$$

Then the components of A_1 are expressed in the components of $\tilde{A}_1$: $p = \tilde{p} + b_1\tilde{p}^2 + \mathcal{O}(\tilde{p}^3)$, $s = \tilde{s} + b_2\tilde{s}^2 + \mathcal{O}(\tilde{s}^3)$. The contribution of A_2 to $\tilde{A}_3$ consists of the third order terms in

$$\begin{pmatrix} 0 & q_{11}(\tilde{p} + b_1\tilde{p}^2)^2/2 + q_{22}(\tilde{s} + b_2\tilde{s}^2)^2/2 \\ r_{11}(\tilde{p} + b_1\tilde{p}^2)^2/2 + r_{22}(\tilde{s} + b_2\tilde{s}^2)^2/2 & 0 \end{pmatrix}.$$

These are

$$\begin{pmatrix} 0 & q_{11}b_1\tilde{p}^3 + q_{22}b_2\tilde{s}^3 \\ r_{11}b_1\tilde{p}^3 + r_{22}b_2\tilde{s}^3 & 0 \end{pmatrix}.$$

The contribution of A_3 to $\tilde{A}_3$ is given by the coefficients of A_3. The contribution of $A_1bA_2 + A_2bA_1$ to $\tilde{A}_3$ follows from

$$\begin{aligned} A_1bA_2 + A_2bA_1 &= \begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} \\ &= \begin{pmatrix} 0 & pb_1q + qb_2s \\ sb_2r + rb_1p & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \tilde{p}b_1\tilde{q} + \tilde{q}b_2\tilde{s} \\ \tilde{s}b_2\tilde{r} + \tilde{r}b_1\tilde{p} & 0 \end{pmatrix} + \mathcal{O}(p,s)^4. \end{aligned}$$

Combining the different terms we see that

$$\begin{aligned} \tilde{A}_3 = A_3 &+ \begin{pmatrix} 0 & q_{11}b_1\tilde{p}^3 + q_{22}b_2\tilde{s}^3 \\ r_{11}b_1\tilde{p}^3 + r_{22}b_2\tilde{s}^3 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \tilde{p}b_1\tilde{q} + \tilde{q}b_2\tilde{s} \\ \tilde{s}b_2\tilde{r} + \tilde{r}b_1\tilde{p} & 0 \end{pmatrix}. \end{aligned}$$

For the action on the third order coefficients to be effective is necessary and sufficient that at least two out of the four coefficients q_{11}, r_{11}, q_{22} and r_{22} are non-zero. For a generic point this action is effective. Hence by normalizing two suitable third order coefficients the structure group reduces to the scalar multiplications. The remaining six third order coefficients are invariants for the surface.

For higher order contact at each order n there are precisely $2(n+1)$ more derivatives. Since the structure group already acted effectively at order 3 (for generic structures) we find at each order precisely $2(n+1)$ additional differential invariants.

Order of contact for characteristic and intersection curves. We already know that the characteristic curves and the intersection curves through a point have the same tangent vectors, i.e., they have the same first order contact. We will show that for every hyperbolic surface the two curves have contact of order at least two.

Using the group action we can arrange that any hyperbolic surface can be written in the normal form

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

with $q = q_{11}p^2/2 + q_{22}s^2/2 + \mathcal{O}(p,s)^3$, $r = r_{11}p^2/2 + r_{22}s^2/2 + \mathcal{O}(p,s)^3$. The intersection curves through the point 0 are determined by

$$0 = \det A = ps - (1/4)\left(q_{11}r_{11}p^4 + (q_{11}r_{22} + q_{22}r_{11})p^2s^2 + q_{22}r_{22}s^4\right) + \mathcal{O}(p,s)^5.$$

The intersection curve tangent to $s = 0$ (the horizontal intersection curve) is of the form

$$s = \frac{1}{4}q_{11}r_{11}p^3 + \mathcal{O}(p^4). \tag{2.17}$$

The other intersection curve (vertical intersection curve) is of the form

$$p = \frac{1}{4} q_{22} r_{22} s^3 + \mathcal{O}(s^4). \tag{2.18}$$

The direction of the characteristics at a point is given by the vector (P, S) that satisfies $\det(P(\partial A/\partial p) + S(\partial A/\partial s)) = 0$. This is equal to

$$\begin{aligned} 0 &= \det\left(P\begin{pmatrix} 1 & q_{11}p \\ r_{11}p & 0 \end{pmatrix} + S\begin{pmatrix} 0 & q_{22}s \\ r_{22}s & 1 \end{pmatrix}\right) \\ &= \det\begin{pmatrix} P & q_{11}pP + q_{22}sS \\ r_{11}pP + r_{22}sS & S \end{pmatrix} \\ &= PS - (q_{11}r_{11}p^2P^2 + (r_{11}q_{22} + r_{22}q_{11})psPS + r_{22}q_{22}s^2S^2). \end{aligned} \tag{2.19}$$

We analyze the characteristic vectors along the intersection curve (2.17). The characteristic vectors are all of the form $S = cP$ for c a small constant depending on the point p, s. We substitute $S = cP$ in the equation (2.19). This yields

$$0 = \left(c - (q_{11}r_{11}p^2 + (r_{11}q_{22} + r_{22}q_{11})psc + r_{22}q_{22}s^2c^2)\right) P^2.$$

Hence on the first intersection curve

$$\begin{aligned} c &= q_{11}r_{11}p^2 + \mathcal{O}(p^4)c + \mathcal{O}(p^6)c^2 \\ &= q_{11}r_{11}p^2 + \mathcal{O}(p)^6. \end{aligned}$$

On the intersection curve the direction of the characteristics is given by (P, S) with

$$S = \left(q_{11}r_{11}p^2 + \mathcal{O}(p)^6\right) P.$$

The tangent vector to the intersection curve follows by differentiation of (2.17) and is equal to the vector (P', S') with

$$S' = \left(\frac{3}{4} q_{11}r_{11}p^2 + \mathcal{O}(p^3)\right) P'.$$

The characteristic curves and intersection curves have at least second order contact. The contact is truly second order if and only if $r_{11}q_{11} \neq 0$. For the vertical characteristic curves and vertical intersection curves the contact is of second order as well, and truly of second order if and only if $q_{22}r_{22} \neq 0$. The second order contact and the factor $3/4$ between the two curves if the contact is truly third order can be seen in Example 2.3.6.

Example 2.3.18 (Invariants for geometrically flat surfaces).

- Consider the geometrically flat surface of type $(2', 3')$ from Example 2.3.5 and Example 2.3.16. The local coordinates are already suited for calculating the invariants. If $\phi(a, b) = q_{11}a^2/2 + q_{12}ab + q_{22}b^2/2 + \mathcal{O}(a, b)^3$, then the second order invariant I is undefined since both the numerator and the denominator are zero.

- In local coordinates define the geometrically flat surface of type $(2', 2')$ by

$$\begin{pmatrix} p & q(s) \\ r(p) & s \end{pmatrix},$$

with $r(p) = r_{11}p^2/2 + \mathcal{O}(p^3)$, $q(s) = q_{22}s^2/2 + \mathcal{O}(s^3)$. The points on the characteristic line $p = p_0$ all have the line $l_1(p_0) = \mathbb{R}(1, 0, p_0, r(p_0))^T$ in common. The points on the characteristic line $s = s_0$ all have the line $l_1(s_0) = \mathbb{R}(0, 1, q(s_0), s_0)^T$ in common. If the surface is generic enough, i.e., $r_{11}, q_{22} \neq 0$, the invariant I is well-defined and equal to zero.

- In local coordinates define the geometrically flat surface of type $(3', 3')$ by

$$\begin{pmatrix} p & q(p) \\ r(s) & s \end{pmatrix},$$

with $r(s) = r_{22}s^2/2 + \mathcal{O}(s)^3$, $q(p) = q_{11}p^2/2 + \mathcal{O}(p)^3$. If the surface is generic enough, i.e., $r_{22}, q_{11} \neq 0$, the invariant I is well-defined and has value ∞. $\oslash$

2.3.4 Moving frames

In this section we will construct a bundle over a hyperbolic surface in the Grassmannian that describes the normalizations we have made in the previous section. The reason for introducing these bundles is that they will allow us to make a connection to the invariants of first order systems in Section 5.3. The reader can skip the remainder of this chapter until that section.

The points in the fiber of the bundle over a point $L \in S$ will provide local coordinates for Grassmannian near L in which the surface is normalized in the way described in the previous section. The idea for this bundle is from McKay [51, Section 4.3] but we give a different presentation. This presentation allows us to translate the invariants on the bundle to the invariants in local coordinates and this allows us to calculate these invariants in examples.

Moving frames at order zero

First we choose basis $e_1^0, e_2^0, e_3^0, e_4^0$ for V. We define $B_V = \mathrm{GL}(V)$ and define the projection $\pi : B_V \to \mathrm{Gr}_2(V) : g \mapsto g^{-1}(\mathbb{R}e_1^0 + \mathbb{R}e_2^0)$. The left action of $\mathrm{GL}(V)$ on B_V induces an action on $\mathrm{Gr}_2(V)$. The stabilizer group of the point $L_0 = \mathbb{R}e_1^0 + \mathbb{R}e_2^0$ is equal to the group H_0, which is the group of matrices

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{d} \end{pmatrix} \in \mathrm{GL}(V), \tag{2.20}$$

with $\tilde{a}, \tilde{d} \in \mathrm{GL}(2, \mathbb{R})$ and $\tilde{b}$ an arbitrary 2×2-matrix. This shows B_V the is a H_0 bundle over the Grassmannian.

Let S be a hyperbolic surface that is embedded as $\iota : S \to \mathrm{Gr}_2(V)$. We can then form the pullback bundle $B_0 = \iota^* B_V$.

$$\begin{tikzcd} B_0 = \iota^* B_V \arrow[r] \arrow[d, "\pi^0"] & B_V \arrow[d, "\pi"] \\ S \arrow[r, hook, "\iota"] & \mathrm{Gr}_2(V) \end{tikzcd}$$

A point in the bundle B_0 is given by a pair $(s, g) \in S \times B_V$ such that $\iota(s) = \pi(g) \in \mathrm{Gr}_2(V)$. Since the map ι is an embedding, the point g already determines the point s and for this reason we will denote a point in B_0 often by the element $g \in B_V$ alone.

Any point $g \in B_0$ defines a frame $e_j = g^{-1}(e_j^0)$ for V. This frame for V introduces local coordinates for $\mathrm{Gr}_2(V)$ in the following way. Use e_1, e_2 as a basis for a 2-plane $L = \mathbb{R}e_1 + \mathbb{R}e_2$ and use e_3, e_4 as a basis for the 2-plane $M = \mathbb{R}e_3 + \mathbb{R}e_4$. Recall the open subset $\mathrm{Gr}_2^0(V, M) \subset \mathrm{Gr}_2(V)$ is isomorphic to the linear maps from L to M. The matrix representations of these maps with respect to the bases e_1, e_2 for L and e_3, e_4 for M provide local coordinates for $\mathrm{Gr}_2^0(V, M)$. We say that these local coordinates are induced by the element $g \in B_V$.

If g is a point in the fiber of B_0 over the point L, then in the local coordinates induced by g the point L corresponds to the zero matrix. Recall that choosing coordinates such that the point L corresponds to the zero matrix is the first step in constructing a normal form for the surface in local coordinates (see Section 2.3.3). We say that the bundle B_0 over S is adapted up to order zero because all points $g \in B_0$ induce local coordinates for S in which the surface is normalized up to order zero in the sense of Section 2.3.3.

The next step in our construction is to construct a bundle B_1 over S such that the points $g \in B_1$ over L induce local coordinates near L in which the surface is adapted up to first order. We will also see that the normalizations in local coordinates correspond to relations between the components of the pullback of the right-invariant Maurer-Cartan on B_V to the bundle B_1.

Remark 2.3.19. McKay [51, Section 4.4] used the relations between the components of the Maurer-Cartan form to *define* the different bundles. In contrast, we show that these relations can be used as an *alternative* definition of the bundles. We use the relations to calculate value of invariants for specific surfaces. We need the calculations in local coordinates to be able to make a precise analysis of the action of $\mathrm{GL}(V)$ on the jets of hyperbolic surfaces. ⊘

Remark 2.3.20 (Frenet frames). The construction described in this section is similar to the construction of Frenet frames for space curves. See for example Ivey and Landsberg [43, pp. 23–26]. We start with a geometric object S that is a submanifold of the base manifold M. The base manifold is realized as the homogeneous space $M = G/H$ for a Lie group G. The group H is the isotropy group of the action of G on M. The points $g \in G$ correspond to points in M by the projection $g \mapsto gH$. But we want that the points $g \in G$ also contain some information about the geometry at the point gH. For example in the case of hyperbolic surfaces the points $g \in G$ provided local coordinates for the hyperbolic surface near the point $gH \in M$. The geometric object is embedded in G/H as $\iota : S \to G/H$. The pullback $\iota^* G$ is a H bundle over S. We then find reductions of this H bundle in such a way that the reduced bundles consist of points g that are adapted to the geometry of S near gH.

In the case of Frenet coframes the group G is ASO(3), the semi-direct product of the translations and rotations in $\mathbb{R}^3$. The elements in ASO(3) are pairs (x, R) with $x \in \mathbb{R}^3$ and $R \in \mathrm{SO}(3)$ a rotation. We define the projection $\mathrm{ASO}(3) \to M = \mathbb{R}^3 : (x, R) \mapsto x$. The action of ASO(3) on itself then defines a transitive action on M. The stabilizer group of the origin in $M = \mathbb{R}^3$ is $H \cong \mathrm{SO}(3)$. The base space M is isomorphic to G/H. The rotation R defines an orthonormal frame at the point x by rotation of the standard basis vectors in $\mathbb{R}^3$. The geometric object is a space curve $c(t)$ parameterized by arc length. The pullback B_0 is the bundle over S for which the fibers over $x = gH \in S$ are the points $\{ (x, R) \in \mathrm{ASO}(3) \mid R \in \mathrm{SO}(3) \}$. A first reduction of B_0 is the reduction to the bundle $B_1 \subset B_0$ for which the fiber $(B_1)_x$ over a point $x = c(0) \in S$ is equal to $\{ (x, R) \in \mathrm{ASO}(3) \mid x = c(0),\ R(e_1) = c'(t) \}$. The bundle B_1 is a SO(2) reduction of B_0. If the curve is generic we can make a second normalization. The vector $c''(t)/|c''(t)|$ is non-zero and perpendicular to $c'(t)$ and we can arrange that $Re_1 = c'(t)$, $Re_2 = c''(t)/|c''(t)|$. This defines a unique frame at the point $c(t)$ which is called the *Frenet frame*.

In the table below the different objects for Frenet frames and our construction for hyperbolic surfaces are given.

	Frenet frames	Hyperbolic surfaces
Base manifold	$\mathbb{R}^3$	$\mathrm{Gr}_2(V)$
Geometric object	Space curve	Hyperbolic surface
G	ASO(3)	$\mathrm{GL}(4, \mathbb{R})$
Isotropy group H	SO(3)	H_0
Geometric interpretation	Frenet frame	Local coordinates
Invariants	Curvature κ, torsion τ	f, g, e

$\oslash$

Remark 2.3.21. We have chosen to work with the unoriented Grassmannian and the group $\mathrm{GL}(V)$. The whole theory can also be done for the oriented planes with some minor modifications. We then have $\widetilde{\mathrm{Gr}}(V)$ as the base space and use the group $\mathrm{GL}^+(V)$ of orientation preserving linear transformations. Since we are mostly interested in the local theory the difference between the two approaches is not essential.

Another choice is related to the fact that the scalar multiplications do not act on $\mathrm{Gr}_2(V)$. This means that we can also choose to work with the group of projective linear transformations $\mathbb{P}\,\mathrm{GL}(V)$. We have chosen to work with $\mathrm{GL}(V)$ because this group has a natural representation as matrix groups. Another option would be to choose a volume form on V. Then the group $\mathrm{SL}(V)$ is isomorphic to $\mathbb{P}\,\mathrm{GL}^+(V)$ and we can use a matrix representation of $\mathrm{SL}(V)$ as a representation for $\mathbb{P}\,\mathrm{GL}^+(V)$. $\oslash$

The Maurer-Cartan form

On the bundle B_V the right-invariant Maurer-Cartan form is defined as $\alpha_R = (\mathrm{d}g)g^{-1}$. This is a 1-form on B_V with values in the Lie algebra $\mathfrak{g} = \mathfrak{gl}(V)$. We will see below that the normalizations in local coordinates correspond to relations between the components of the pullback of the Maurer-Cartan form to the bundles B_0 and B_1. It is convenient to write the Maurer-Cartan form as

$$\alpha_R = (\mathrm{d}g)g^{-1} = \begin{pmatrix} \xi & \eta \\ \vartheta & \zeta \end{pmatrix}, \tag{2.21}$$

for 1-forms $\xi, \eta, \vartheta, \zeta$ valued in the space of 2×2-matrices. Using the decomposition of 2×2-matrices in Section 2.2.4 we have D-valued 1-forms $\eta', \eta'', \ldots, \zeta', \zeta''$.

The right-invariant Maurer-Cartan form satisfies the structure equations $\mathrm{d}\alpha_R = \alpha_R \wedge \alpha_R$. This implies the following set of structure equations for the components

$$\mathrm{d}\begin{pmatrix} \xi' \\ \xi'' \\ \eta' \\ \eta'' \\ \vartheta' \\ \vartheta'' \\ \zeta' \\ \zeta'' \end{pmatrix} = \begin{pmatrix} \xi'' \wedge \xi''^F + \eta' \wedge \vartheta' + \eta'' \wedge \vartheta''^F \\ \xi' \wedge \xi'' + \xi'' \wedge \xi'^F + \eta' \wedge \vartheta'' + \eta'' \wedge \vartheta'^F \\ \xi' \wedge \eta' + \xi'' \wedge \eta''^F + \eta' \wedge \zeta' + \eta'' \wedge \zeta''^F \\ \xi' \wedge \eta'' + \xi'' \wedge \eta'^F + \eta' \wedge \zeta'' + \eta'' \wedge \zeta'^F \\ \vartheta' \wedge \xi' + \zeta' \wedge \vartheta' + \vartheta'' \wedge \xi''^F + \zeta'' \wedge \vartheta''^F \\ \vartheta' \wedge \xi'' + \zeta'' \wedge \vartheta'^F + \vartheta'' \wedge \xi'^F + \zeta' \wedge \vartheta'' \\ \vartheta' \wedge \eta' + \vartheta'' \wedge \eta''^F + \zeta'' \wedge \zeta''^F \\ \vartheta' \wedge \eta'' + \vartheta'' \wedge \eta'^F + \zeta' \wedge \zeta'' + \zeta'' \wedge \zeta'^F \end{pmatrix}. \tag{2.22}$$

The right-invariant Maurer-Cartan form transforms under the left multiplications by the adjoint action. We have $(L_h^* \alpha_R)_g = \mathrm{Ad}(h) \circ (\alpha_R)_g$.

Since $\dim B_0 = 14$ there will be relations between the 1-forms $\xi, \eta, \vartheta, \zeta$ on the pullback bundle. Later we will use the structure group H_0 of B_0 over S to normalize these relations. Note that the four 1-forms $\vartheta', \vartheta'^F, \vartheta'', \vartheta''^F$ vanish on the fibers of the bundle $B_0 \to S$ and hence there must be relations between these forms since S is only two-dimensional.

Moving frames at order one

We have constructed the principal H_0 bundle B_0 over a hyperbolic surface S. This bundle was adapted to the geometry of S in the sense that in the local coordinates induced by a point $g \in B_0$ the point $s \in S$ is given by the zero matrix. The next step is to reduce the bundle B_0 to a bundle B_1 that is adapted to the surface up to order one.

Let the surface be given in local coordinates as

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

and assume that, at the point L given by $A = 0$, the coordinates are adapted up to first order. This means that $q = \mathcal{O}(p, s)^2$ and $r = \mathcal{O}(p, s)^2$. In other words the first order part of A at L, given by $\mathrm{d}A$, is diagonal.

A section of the bundle B_0 is given by

$$S \to B_0 : A \mapsto g_0 = \begin{pmatrix} I & 0 \\ -A & I \end{pmatrix}.$$

Since the structure group for B_0 is the group H_0 (see (2.20)) the points in the bundle B_0 can be parameterized as hg_0 for $h \in H_0$. Let us calculate the pullback of the right-invariant Maurer-Cartan form on B_V to B_0. The Maurer-Cartan form at a point $hg_0 \in B_0$ is given by

$$\begin{aligned}(\alpha_R)_{hg_0} &= (\mathrm{d}hg_0)(hg_0)^{-1} = (\mathrm{d}h)h^{-1} + h(\mathrm{d}g_0)g_0^{-1}h^{-1} \\ &= (\mathrm{d}h)h^{-1} + \begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{d} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\mathrm{d}A & 0 \end{pmatrix} \begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{d} \end{pmatrix}^{-1} \\ &= (\mathrm{d}h)h^{-1} + \begin{pmatrix} -\tilde{b}(\mathrm{d}A)\tilde{a}^{-1} & \tilde{b}(\mathrm{d}A)\tilde{a}^{-1}\tilde{b}\tilde{d}^{-1} \\ -\tilde{d}(\mathrm{d}A)\tilde{a}^{-1} & \tilde{d}(\mathrm{d}A)\tilde{a}^{-1}\tilde{b}\tilde{d}^{-1} \end{pmatrix}.\end{aligned} \tag{2.23}$$

The term $(\mathrm{d}h)h^{-1}$ has no contribution to the bottom left part ϑ of α_R. Hence the action of h on the bottom left part is by a conformal action. We can always arrange by a suitable choice of $\tilde{a}, \tilde{d}$ that $\vartheta'' = 0$. The stabilizer group of $\vartheta'' = 0$ is given by the subgroup H_1 consisting of matrices

$$h_1 = \begin{pmatrix} a & \tilde{b} \\ 0 & d \end{pmatrix} \in \mathrm{GL}(4, \mathbb{R}) \tag{2.24}$$

for which $a, d \in D$ or $a, d \in LD$. So either a, d are both diagonal or both anti-diagonal. We define the bundle B_1 as the subbundle of B_0 for which $\vartheta'' = 0$. The calculation above shows that B_1 is a H_1 reduction of B_0.

Remark 2.3.22. The condition that $(\tilde{d}\vartheta\tilde{a}^{-1})'' = 0$ if $\vartheta'' = 0$ is precisely that $a'' = d'' = 0$ or $a' = d' = 0$. Indeed, let $\tilde{a}^{-1} = \tilde{k}$. Then

$$\tilde{d}\vartheta\tilde{a}^{-1} = \tilde{d}\vartheta\tilde{k} = d'\vartheta'k' + d''\vartheta'^F k''^F + d'\vartheta'k''L + d''\vartheta'^F k'L.$$

Since ϑ' and ϑ'^F are linearly independent we find the two equations $d'k'' = 0$ and $d''k' = 0$. Since $\tilde{a}$ and $\tilde{d}$ are invertible this implies either $d'' = k'' = a'' = 0$ or $d' = k' = a' = 0$. ⊘

Remark 2.3.23. McKay also defines a bundle B_1 for the elliptic surfaces, but does not mention the fact that the full structure group that preserves the equation $\vartheta'' = 0$ has two connected components. In McKay [51, p. 28] he defines a bundle B_1 which is the elliptic equivalent of one of the connected components of our bundle B_1. The fact that he only uses the connected component might be related to the orientation that he uses, but he does not explicitly mention this. This omission also occurs in McKay [52, p. 14] (in this article the bundle has the name B_2, which is confusing with the notation in McKay's thesis [51] and this dissertation). ⊘

We have defined B_1 by the condition $\vartheta'' = 0$. If the surface is given in local coordinates induced by $g \in B_0$ by the 2×2-matrix A and $\mathrm{d}A$ is diagonal at g, then equation (2.23) shows that $g \in B_1$. All other points $\tilde{g}$ in the same fiber of B_1 as g are related by an element of H_1. The local coordinates induced by $\tilde{g}$ are then related to the local coordinates induced by g by a transformation by an element of H_1. From the discussion about the normal form in local coordinates for a hyperbolic surface at order one starting on page 61 it follows that the transformations in H_1 are precisely the transformations that leave invariant the condition that $\mathrm{d}A$ is diagonal. Hence for all $\tilde{g}$ in the fiber of B_1 the surface in local local coordinates induced by $\tilde{g}$ is normalized up to order one in the sense of Section 2.3.3.

Moving frames at order two

For any point in B_1 the pullback of Maurer-Cartan form takes values in $\mathfrak{g}$. Since B_1 has dimension $2 + 8 = 10$ the image of the tangent space of B_1 under α_R is a 10-dimensional subspace of $\mathfrak{g}$. The Maurer-Cartan form of the structure group H_1 is of the form

$$(\mathrm{d}h)h^{-1} = \begin{pmatrix} \mathrm{d}a & \mathrm{d}\tilde{b} \\ 0 & \mathrm{d}d \end{pmatrix} \begin{pmatrix} a & \tilde{b} \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} (\mathrm{d}a)a^{-1} & -a^{-1}(\mathrm{d}a)\tilde{b}d^{-1} + (\mathrm{d}\tilde{b})d^{-1} \\ 0 & (\mathrm{d}d)d^{-1} \end{pmatrix}.$$

The tangent space to the fiber of the projection $B_1 \to S$ at a point is mapped by the Maurer-Cartan form to the eight-dimensional Lie algebra $\mathfrak{h}_1$ of H_1 in $\mathfrak{g}$. So the tangent space of B_1 is mapped by the Maurer-Cartan form to a 2-dimensional linear subspace in $\mathfrak{g}/\mathfrak{h}_1$. If we move in the direction transversal to the fibers, then $\vartheta' \neq 0$. Also on B_1 the form $\vartheta'' = 0$. This implies that the 2-dimensional linear space in $\mathfrak{g}/\mathfrak{h}_1$ is determined by relations

$$\xi'' = f\vartheta' + e_1(\vartheta')^F, \quad \zeta'' = e_2\vartheta' + g(\vartheta')^F,$$

with f, g, e_1 and e_2 all $\mathbb{D}$-valued functions on B_1.

We will calculate the consequences of the reduction from B_0 to B_1 for the structure equations (2.22). Since $\vartheta'' = 0$ we find directly that $0 = \mathrm{d}\vartheta'' = \vartheta' \wedge \xi'' + \zeta'' \wedge \vartheta'^F$. By Cartan's lemma (Lemma 1.2.12) we conclude that

$$\begin{pmatrix} \xi'' \\ \zeta'' \end{pmatrix} = \begin{pmatrix} f & e \\ -e & g \end{pmatrix} \begin{pmatrix} \vartheta' \\ \vartheta'^F \end{pmatrix},$$

for functions f, g, h valued in the hyperbolic numbers. So $e = e_1 = e_2$. This equality is a consequence of the equality of mixed second order derivatives in local coordinates. Let h_1 be the matrix from (2.24) with $a, d \in D$. The action by left multiplication on B_1 acts on the Maurer-Cartan form by the adjoint representation.

$$\begin{aligned}
&\mathrm{Ad}(h_1)\alpha_R \\
&= \begin{pmatrix} a & \tilde{b} \\ 0 & d \end{pmatrix} \begin{pmatrix} \xi & \eta \\ \vartheta & \zeta \end{pmatrix} \begin{pmatrix} a^{-1} & -a^{-1}\tilde{b}d^{-1} \\ 0 & d^{-1} \end{pmatrix} \\
&= \begin{pmatrix} a & \tilde{b} \\ 0 & d \end{pmatrix} \begin{pmatrix} \xi' + \xi''L & \eta' + \eta''L \\ \vartheta' & \zeta' + \zeta''L \end{pmatrix} \begin{pmatrix} a^{-1} & -a^{-1}\tilde{b}d^{-1} \\ 0 & d^{-1} \end{pmatrix} \\
&= \begin{pmatrix} a\xi' + a\xi''L + \tilde{b}\vartheta' & a\eta' + a\eta''L + \tilde{b}\zeta' + \tilde{b}\zeta''L \\ d\vartheta' & d\zeta' + d\zeta''L \end{pmatrix} \begin{pmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{pmatrix} \\
&= \begin{pmatrix} a\xi'a^{-1} + a\xi''La^{-1} + \tilde{b}\vartheta'a^{-1} & * \\ d\vartheta'a^{-1} & d\vartheta'(-a^{-1}\tilde{b}d^{-1}) + c\zeta'd^{-1} + c\zeta''Ld^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \xi' + a{a^{-1}}^F\xi''L + \tilde{b}a^{-1}\vartheta' & * \\ da^{-1}\vartheta' & d\vartheta'(-a^{-1}bd^{-1}) + \zeta' + d{d^{-1}}^F\zeta''L \end{pmatrix}.
\end{aligned}$$

We have $\xi'' = f\vartheta' + e\vartheta'^F$ and $\tilde{\vartheta}' = da^{-1}\vartheta'$. The transformed components are

$$\begin{aligned}
\tilde{\xi}'' &= aa^{-1^F}\xi'' + b''a^{-1^F}\vartheta'^F \\
&= aa^{-1^F}(f\vartheta' + e\vartheta'^F) + b''a^{-1^F}\vartheta'^F \\
&= aa^{-1^F} fad^{-1}\tilde{\vartheta}' + aa^{-1^F} ea^F d^{-1^F}(\tilde{\vartheta}')^F + b''a^{-1^F} ad^{-1}\tilde{\vartheta}' \\
&= a^2 a^{-1^F} d^{-1} f\tilde{\vartheta}' + (ae + b'')d^{-1^F}(\tilde{\vartheta}')^F, \\
\tilde{\zeta}'' &= -da^{-1}b''d^{-1^F}\vartheta' + dd^{-1^F}\zeta'' \\
&= -da^{-1}b''d^{-1^F}\vartheta' + dd^{-1^F}(-h\vartheta' + g\vartheta'^F) \\
&= (-da^{-1}b''d^{-1^F} - dd^{-1^F}e)\vartheta' + dd^{-1^F}g\vartheta'^F \\
&= (-da^{-1}b''d^{-1^F} - dd^{-1^F}e)ad^{-1}\tilde{\vartheta}' + dd^{-1^F}ga^F d^{-1^F}(\tilde{\vartheta}')^F \\
&= (-b''d^{-1^F} - ad^{-1^F}e)\tilde{\vartheta}' + dd^{-2^F}ga^F(\tilde{\vartheta}')^F \\
&= -(b'' + ae)d^{-1^F}\tilde{\vartheta}' + a^F dd^{-2^F}g(\tilde{\vartheta}')^F.
\end{aligned}$$

From this we can read off that f, g, e transform as

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} f \\ g \\ e \end{pmatrix} = \begin{pmatrix} a^2(a^{-1})^F d^{-1} f \\ a^F d(d^{-2})^F g \\ (ae + b'')(d^{-1})^F \end{pmatrix}. \tag{2.25}$$

We can recognize these transformations as the transformation rules (2.14) for the second order coefficients for a hyperbolic surface in local coordinates. We will prove that the second order coefficients in local coordinates indeed correspond to the functions f, g and e.

Suppose a hyperbolic surface is given in local coordinates by matrices

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

We assume that at the origin the surface is adapted and so we can write $q = q_{11}p^2/2 + q_{12}ps + q_{22}s^2/2 + \mathcal{O}(p,s)^3$, $r = r_{11}p^2/2 + r_{12}ps + r_{22}s^2/2 + \mathcal{O}(p,s)^3$. We want to calculate the bundle B_1 so that we can calculate the relations between ξ'', ζ'' and ϑ'. Since the group H_1 acts transitively on the fibers of B_1, it is enough to find a section of the bundle B_1. Calculating this section exactly is difficult, but for our purposes it will be enough to find this section up to order 2 at a special point. We will prove the equalities we need at the special point and the result then follows from the transformation rules.

A section of B_0 over the surface is given by

$$g_0 = \begin{pmatrix} I & 0 \\ -A & I \end{pmatrix}.$$

We can make this into a section of B_1 by multiplying with a suitable element of H_0. We take

$$h = \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix} \quad \text{with} \quad \tilde{a} = \begin{pmatrix} 1 & c_1 \\ c_2 & 1 \end{pmatrix},\ \tilde{d} = \begin{pmatrix} 1 & c_3 \\ c_4 & 1 \end{pmatrix}.$$

Then $g_1 = hg_0$ is a section of B_0. The pullback of the Maurer-Cartan form over this section is given by

$$
\begin{aligned}
(\alpha_R)_{g_1} &= \mathrm{d}(hg_0)(hg_0)^{-1} = (\mathrm{d}h)h^{-1} + h(\mathrm{d}g_0)g_0^{-1}h^{-1} \\
&= (\mathrm{d}h)h^{-1} + \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\mathrm{d}A & 0 \end{pmatrix} \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix}^{-1} \\
&= (\mathrm{d}h)h^{-1} + \begin{pmatrix} 0 & 0 \\ -\tilde{d}(\mathrm{d}A)\tilde{a}^{-1} & 0 \end{pmatrix}.
\end{aligned}
$$

The section g_1 of B_0 is a section of B_1 if the lower left block is diagonal. The term $(\mathrm{d}h)h^{-1}$ has no contribution to the lower left block. The term $\tilde{d}(\mathrm{d}A)\tilde{a}^{-1}$ is equal to

$$
\begin{aligned}
\tilde{d}(\mathrm{d}A)\tilde{a}^{-1} &= \begin{pmatrix} 1 & c_1 \\ c_2 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}p & \begin{array}{l} q_{11}p\mathrm{d}p + q_{22}s\mathrm{d}s \\ \quad + q_{12}(s\mathrm{d}p + p\mathrm{d}s) \end{array} \\ \begin{array}{l} r_{11}p\mathrm{d}p + r_{22}s\mathrm{d}s \\ \quad + r_{12}(s\mathrm{d}p + p\mathrm{d}s) \end{array} & \mathrm{d}s \end{pmatrix} \begin{pmatrix} 1 & c_3 \\ c_4 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \mathrm{d}p + \mathcal{O}(p,s) & \begin{array}{l} q_{11}p\mathrm{d}p + q_{12}(s\mathrm{d}p + p\mathrm{d}s) \\ + q_{22}s\mathrm{d}s + c_1\mathrm{d}p + c_3\mathrm{d}s \\ + \mathcal{O}(p,s)^2 \end{array} \\ \begin{array}{l} r_{11}p\mathrm{d}p + r_{12}(s\mathrm{d}p + p\mathrm{d}s) \\ + r_{22}s\mathrm{d}s + c_4\mathrm{d}p + c_2\mathrm{d}s \\ + \mathcal{O}(p,s)^2 \end{array} & \mathrm{d}s + \mathcal{O}(p,s) \end{pmatrix}.
\end{aligned}
$$

By taking

$$
\begin{aligned}
c_1 &= -(q_{11}p + q_{12}s) + \mathcal{O}(p,s)^2, \quad c_3 = -(q_{12}p + q_{22}s) + \mathcal{O}(p,s)^2, \\
c_4 &= -(r_{11}p + r_{12}s) + \mathcal{O}(p,s)^2, \quad c_2 = -(r_{12}p + r_{22}s) + \mathcal{O}(p,s)^2,
\end{aligned}
$$

we find a section $g_1 = hg_0$ of the bundle B_1.

Then the general element of B_1 is given by h_1g_1 for $h_1 \in H_1$. The Maurer-Cartan form is given by

$$
(\alpha_R)_{h_1g_1} = \mathrm{d}(h_1g_1)(h_1g_1)^{-1} = (\mathrm{d}h_1)h_1^{-1} + h_1(\mathrm{d}g_1)(g_1^{-1})h_1^{-1}.
$$

Here

$$
\begin{aligned}
\mathrm{d}(g_1)(g_1)^{-1} &= \begin{pmatrix} \mathrm{d}\tilde{a} & 0 \\ (\mathrm{d}\tilde{d})A + \tilde{d}(\mathrm{d}A) & \mathrm{d}\tilde{d} \end{pmatrix} g_1^{-1} \\
&= \begin{pmatrix} (\mathrm{d}\tilde{a})\tilde{a}^{-1} & 0 \\ (\mathrm{d}\tilde{d})A\tilde{a}^{-1} + \tilde{d}(\mathrm{d}A)\tilde{a}^{-1} - (\mathrm{d}\tilde{d})\tilde{d}^{-1}\tilde{a}A\tilde{a}^{-1} & (\mathrm{d}\tilde{d})\tilde{d}^{-1} \end{pmatrix}.
\end{aligned}
$$

In the origin we have $A = 0$ and $\mathrm{d}A = \begin{pmatrix} \mathrm{d}p & 0 \\ 0 & \mathrm{d}s \end{pmatrix}$. Hence $\vartheta' = (\mathrm{d}p, \mathrm{d}s)^T$, $\vartheta'' = 0$. Also

$$\begin{aligned}
\tilde{a}^{-1} &= \begin{pmatrix} 1 & -q_{11}p - q_{12}s \\ -r_{12}p - r_{22}s & 1 \end{pmatrix}^{-1} + \mathcal{O}(p,s)^2 \\
&= \begin{pmatrix} 1 & q_{11}p + q_{12}s \\ r_{12}p + r_{22}s & 1 \end{pmatrix} + \mathcal{O}(p,s)^2, \\
(\mathrm{d}\tilde{a})\tilde{a}^{-1} &= \begin{pmatrix} 0 & -q_{11}\mathrm{d}p - q_{12}\mathrm{d}s \\ -r_{12}\mathrm{d}p - r_{22}\mathrm{d}s & 0 \end{pmatrix} + \mathcal{O}(p,s)^2.
\end{aligned}$$

The component ξ'' of the Maurer-Cartan form is given by $((\mathrm{d}\tilde{a})\tilde{a}^{-1})''$ since the term $(\mathrm{d}h_1)h_1^{-1}$ has no contribution and at the special point h_1 is the identity matrix. Hence

$$\xi'' = \begin{pmatrix} -q_{11}\mathrm{d}p \\ -r_{22}\mathrm{d}s \end{pmatrix} + \begin{pmatrix} -q_{12}\mathrm{d}s \\ -r_{12}\mathrm{d}p \end{pmatrix} = \begin{pmatrix} -q_{11} \\ -r_{22} \end{pmatrix} \vartheta' + \begin{pmatrix} -q_{12} \\ -r_{12} \end{pmatrix} (\vartheta')^F.$$

In a similar way we can calculate that

$$\zeta'' = \begin{pmatrix} q_{12} \\ r_{12} \end{pmatrix} \vartheta' + \begin{pmatrix} -q_{22} \\ -r_{11} \end{pmatrix} (\vartheta')^F.$$

This means that at g_1 we the values of f, g, e correspond to the second order jets of the surface in local coordinates.

Higher order

Since a is invertible we can transform h into zero by choosing a unique element b''. This reduces the bundle B_1 to a new bundle, which we denote by B_S. The structure group of the bundle is the parabolic subgroup of $\mathrm{GL}(2, \mathbb{D})$ given by matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

On B_S we have $\xi'' = f\vartheta'$, $\zeta'' = g(\vartheta')^F$. The functions f, g are functions on B_S that transform under the action given in formula (2.25). The functions to not descent to the surface S, hence they are not invariants. From the action we can see that the functions f, g do define a pair of relative invariants. The expression $|f|^2/|g|^2$ is invariant under the structure group and is equal to the invariant I.

Example 2.3.24. Let K be the standard hyperbolic structure on $\mathbb{R}^4$. The standard basis for $\mathbb{R}^4$ we denote by e_j^0 and we define $L_0 = \mathbb{R}e_1^0 + \mathbb{R}e_2^0$. The group $\mathbb{P}\,\mathrm{GL}(2, \mathbb{D})$ as a subgroup of $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$ acts transitively on the set of all hyperbolic lines. It is not difficult to see that $G = \mathbb{P}\,\mathrm{GL}(2, \mathbb{D})$ is precisely the bundle B_S over the hyperbolic surface $S = \mathrm{Gr}_2(V, K)$. We can introduce local coordinates a, b, c, d for G, where $a, b, c, d \in D$ and $\Delta = ad - bc \in D^*$ and the elements in G are represented by the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The pullback of the right-invariant Maurer-Cartan form on GL(4, $\mathbb{R}$) to G is

$$\begin{pmatrix} \xi & \eta \\ \vartheta & \zeta \end{pmatrix} = \Delta^{-1} \begin{pmatrix} d\mathrm{d}a - c\mathrm{d}b & -b\mathrm{d}a + a\mathrm{d}b \\ d\mathrm{d}c - c\mathrm{d}d & -b\mathrm{d}c + a\mathrm{d}d \end{pmatrix}.$$

We see that for this pullback we have the relations $\vartheta'' = 0$, $\xi'' = \zeta'' = 0$. Hence the projection $G \to \mathrm{Gr}_2(\mathbb{R}^4, K) : g \mapsto g^{-1}(L_0)$ is a bundle over $\mathrm{Gr}_2(\mathbb{R}^4, K)$ that is a subbundle of B_1. The relations $\xi'' = \zeta'' = 0$ show that the invariants f, g and e are zero for the surface. $\oslash$

Chapter 3

Geometry of partial differential equations

In this chapter we introduce the two main classes of partial differential equations that are the subject of this dissertation. The classes are the determined first order systems of partial differential equations for two unknown functions of two variables and the second order scalar partial differential equations in the plane. These two classes of equations have a very similar geometric structure and this is precisely the reason that we can develop the theory for these systems in parallel.

Small variations of these systems, such as introducing more dependent or independent variables, considering under- or overdetermined systems or higher order equations can lead to systems of partial differential equations for which the geometry is completely different from the geometry in the two systems mentioned above. This makes it difficult to apply the theory to be developed in this dissertation to these other types of systems.

3.1 Ordinary differential equations

We start with describing the geometry of ordinary differential equations to show the methods used. Consider the ordinary differential equation

$$z'' = F(x, z, z'). \tag{3.1}$$

The solutions to this equation are functions $z(x)$ that satisfy $z''(x) = F(x, z(x), z'(x))$. The graph of a 2-jet of a solution is a submanifold of the second order jet bundle $\mathrm{J}^2(\mathbb{R})$. On $\mathrm{J}^2(\mathbb{R})$ we have coordinates $x, z, p = z_x, h = z_{xx}$ and the two contact forms $\theta^1 = \mathrm{d}z - p\mathrm{d}x$, $\theta^2 = \mathrm{d}p - h\mathrm{d}x$. To each function f we associate the submanifold

$$S_f = \{\, (x, f(x), f'(x), f''(x)) \in \mathrm{J}^2(\mathbb{R}^2) \mid x \in \mathbb{R}\,\}.$$

The submanifolds S_f are integral manifolds of the Pfaffian system $I = \mathrm{span}(\theta^1, \theta^2)$.

If f is a solution to the differential equation (3.1), then S_f is a submanifold of the hypersurface $M \subset \mathrm{J}^2(\mathbb{R})$ defined by $h = F(x, z, p)$. Conversely, every integral manifold of the Pfaffian system I restricted to M for which $\mathrm{d}x \neq 0$ is locally the graph of a solution. The correspondence between solutions of (partial) differential equations and integral manifolds of a Pfaffian system (or more general an exterior differential ideal) will be used many times.

3.2 Second order scalar equations

We consider a second order partial differential equation for the unknown function z and the independent variables x, y. We introduce the coordinates $x, y, z, p = z_x, q = z_y, r = z_{xx}, s = z_{xy}, t = z_{yy}$ for the second order contact bundle $\mathrm{J}^2(\mathbb{R}^2, \mathbb{R})$. The use of p, q and r, s, t for the first and second order derivatives, respectively, was introduced by Monge. The coordinates x, y, z, p, q, r, s, t are called *Monge coordinates* or *classical coordinates*. The most general form of such an equation is given by

$$F(x, y, z, p, q, r, s, t) = 0. \tag{3.2}$$

We require that that $(F_r, F_s, F_t) \neq 0$, so that (3.2) defines a truly second order equation. If $F_r \neq 0$, then we can (locally) solve for r and rewrite the equation as

$$r = \rho(x, y, z, p, q, s, t). \tag{3.3}$$

If $F_r = 0$ at a point, then we can either solve for one of the other second order variables or we can make a coordinate transformation $x \mapsto x + y$, $y \mapsto x - y$ or $x \mapsto y$, $y \mapsto x$ such that in the new coordinates $F_r \neq 0$.

Let us analyze the geometry of such an equation. On the second order jet bundle the equation (3.2) or (3.3) defines a hypersurface M. On the second order contact bundle we have the contact forms

$$\begin{aligned} &\theta^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\ &\theta^1 = \mathrm{d}p - r\mathrm{d}x - s\mathrm{d}y, \quad \theta^2 = \mathrm{d}q - s\mathrm{d}x - t\mathrm{d}y. \end{aligned} \tag{3.4}$$

These contact forms pull back to contact forms on M, which will also be denoted by θ^j. Assume that $F_r \neq 0$ and that we can solve for r as $r = \rho(x, y, z, p, q, s, t)$. On the hypersurface M the contact forms are

$$\begin{aligned} &\theta^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\ &\theta^1 = \mathrm{d}p - \rho\mathrm{d}x - s\mathrm{d}y, \quad \theta^2 = \mathrm{d}q - s\mathrm{d}x - t\mathrm{d}y. \end{aligned} \tag{3.5}$$

The solutions of the partial differential equation (3.2) are locally in one-to-one correspondence with the integral manifolds of the Pfaffian system generated by the forms (3.5) that satisfy the independence condition $\mathrm{d}x \wedge \mathrm{d}y \neq 0$.

The distribution $\mathcal{V}$ dual to the contact forms is spanned by

$$\begin{aligned} &\partial_x + p\partial_z + \rho\partial_p + s\partial_q, \quad \partial_y + q\partial_z + s\partial_p + t\partial_q, \\ &\partial_s, \quad \partial_t. \end{aligned} \tag{3.6}$$

A simple calculation shows that

$$C(\mathcal{V}) = \text{span}\,(0)\,, \quad \mathcal{V}' = \text{span}\left(\mathcal{V}, \partial_p, \partial_q\right),$$
$$C(\mathcal{V}') = \text{span}\,(\partial_s, \partial_t)\,, \quad \mathcal{V}'' = \text{span}\left(\mathcal{V}', \partial_z\right) = TM.$$

These expressions are only expressions in local coordinates, but since ρ was general we can see that all systems that come from second order scalar equations have a rank 4 distribution $\mathcal{V}$ with the properties that rank $\mathcal{V}' = 6$, rank $\mathcal{V}'' = 7$, rank $C(\mathcal{V}') = 2$, $C(\mathcal{V}') \subset \mathcal{V}$. These conditions on $\mathcal{V}$ describe the entire geometry of the equation. We will see in Section 4.1 that distributions with these properties can locally be written as a second order equation.

3.3 First order systems

Consider a first order system of partial differential equations in the independent variables $x^1, \ldots, x^n$ and dependent variables $u^1, \ldots, u^s$ defined by the equations

$$F^\lambda(x, u, p) = 0 \quad \lambda = 1, \ldots, c. \tag{3.7}$$

Here p^i_j stands for the first order derivative $\partial u^i/\partial x^j$. We can define this system in a coordinate invariant way as follows. Let B be an open subset of $\mathbb{R}^n \times \mathbb{R}^s$ and consider the Grassmann bundle $\text{Gr}_n(TB)$ of n-planes over B. If we assume the equations (3.7) are of constant maximal rank c and we can solve for the first order variables p, then the equations define a codimension c submanifold M of the Grassmann bundle and the canonical projection $\pi : M \to B$ is a submersion. On the Grassmann bundle $\text{Gr}_2(TB)$ we have the contact system I, which is generated by the contact forms $\theta^i = \mathrm{d}u^i - p^i_j \mathrm{d}x^j$, $i = 1, \ldots, s$. The pullbacks of the contact forms define the contact system on M. This contact system is of constant rank and defines the contact distribution $\mathcal{V} = I^\perp$. Solutions $u(x)$ of the system (3.7) correspond to integral manifolds of the contact system on M.

We will specialize to the case $n = s = c = 2$. For such first order systems we will write x, y for the independent variables, u, v for the dependent variables and $p = u_x, q = u_y, r = v_x, s = v_y$ for the first order derivatives. The equation manifold M has dimension 6. From this point on we adopt the convention that, unless stated otherwise, a *first order system* is a first order system of two partial differential equations in two independent and two dependent variables. The definition of such a system can be given by equations in local coordinates, in terms of a distribution on a 6-dimensional manifold or in terms of a codimension 2 Pfaffian system.

Any codimension 2 first order system is defined by two equations $F^1 = 0$, $F^2 = 0$ and can be characterized as follows. If we define $\omega^1 = \mathrm{d}x$, $\omega^2 = \mathrm{d}y$, then the structure equations for the contact forms are

$$\mathrm{d}\theta \equiv \begin{pmatrix} \pi^1_1 & \pi^1_2 \\ \pi^2_1 & \pi^2_2 \end{pmatrix} \wedge \omega \mod I.$$

Notice that any system satisfies $\mathrm{d}\theta \equiv 0 \mod J$, $J = \text{span}(I, \omega^1, \omega^2)$. Hence the graphs of solutions to the partial differential equation correspond to integral manifolds of the linear

Pfaffian system (I, J). Since $\mathrm{d}\omega \equiv 0 \mod J$, the distribution dual to J is an integrable rank two distribution. The leaves of this distribution are precisely the fibers of the projection $M \to B$. The contact distribution $\mathcal{V}$ on M is the distribution dual to I. The Lie brackets modulo the subbundle define a tensor $\mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}$. We will see later that for elliptic and hyperbolic systems this tensor is non-degenerate (the precise definition of non-degenerate is given in Appendix A.1). We will use these properties to define a generalization of a first order system in which there is no distinguished set of independent or dependent variables.

Definition 3.3.1. A *generalized first order system* under *contact geometry* is a smooth manifold M of dimension 6 with a rank 4 distribution $\mathcal{V}$ such that the Lie brackets modulo the subbundle is a non-degenerate map $\mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}$. A *generalized first order system* under *point geometry* is a generalized first order system $(M, \mathcal{V})$ with an integrable distribution $\mathcal{U} \subset \mathcal{V}$.

A *contact transformation* of a first order system is a diffeomorphism leaving invariant $\mathcal{V}$. A *point transformation* is a diffeomorphism leaving invariant both $\mathcal{V}$ and $\mathcal{U}$.

The discussion above shows that any elliptic or hyperbolic first order system defines a generalized first order system under contact geometry and a generalized first order system under point geometry as well. In Section 4.6.1 we will see that any analytic generalized first order system can be written locally as an elliptic or hyperbolic system of partial differential equations. The parabolic first order systems can also be formulated on a 6-dimensional manifold with rank 4 distribution. The author is not aware of conditions on $\mathcal{V}$ that guarantee that $(M, \mathcal{V})$ is locally equivalent to a first order parabolic system. For some non-generic distributions (for example if the distribution is integrable), there are no corresponding systems of equations.

The definition (3.3.1) of a generalized first order system under point geometry is very similar to the definition in McKay [51, p. 37] of almost generalized Cauchy-Riemann equations. The only differences are that McKay only defines the systems for equations of elliptic type (for the type of first order system see Section 4.6) and that he introduces an orientation for the manifold M.

Remark 3.3.2. A structure theory of first order systems in terms of vector fields was already introduced by Vessiot [67]. Dual to the formulation in vector fields there is a formulation in differential forms. For first order elliptic systems McKay [51] has developed the structure theory in great detail. The structure theory using differential forms and vector fields is developed by Vassiliou for hyperbolic systems [65, 66]. $\oslash$

Chapter 4

Contact distributions for partial differential equations

In this chapter we will formulate the structure theory of second order scalar partial differential equations in the plane in terms of contact distributions. Vessiot [67, p. 307] already gave a beautiful geometric characterization of these equations, see Theorem 4.1.2. The papers by Vessiot have been summarized by Stormark [64, Section 5.2, Chapter 11] and Duistermaat [25, Section 3.6]. Here we give our own formulation and extend the structure theory given by these authors.

The theory for first order systems is very similar, but we will not explicitly derive the results for these systems. In Section 4.6 the theory for first order systems is discussed. As an application of the theory we will construct a framing on the equation manifold of a first order system that is invariant under general contact transformations. This framing can then be used to define invariants for the system. This section is complementary to the theory in Chapter 5 and 6. In those chapters differential forms and the method of equivalence are used to develop a structure theory.

4.1 The contact distribution

Let $F(x^i, z, p^i, h^{ij}) = 0$ be a second order scalar partial differential equation in n independent variables $x^1, \dots, x^n$. The function F defines a hypersurface in the second order contact bundle Q. This second order contact bundle has local coordinates x^i, z, p^i, h^{ij}. The contact forms on Q are generated by

$$\theta^0 = \mathrm{d}z - p^j \mathrm{d}x^j, \quad \theta^i = \mathrm{d}p^i - h^{ij}\mathrm{d}x^j, \quad i = 1, \dots, n.$$

The distribution $\mathcal{V}$ dual to the contact forms has dimension $n + n(n+1)/2$.

Definition 4.1.1. Let M be a smooth manifold and $\mathcal{V}$ a smooth distribution on M. The pair $(M, \mathcal{V})$ is called a *Vessiot system* in n variables if it satisfies the following conditions:

- The codimension of $\mathcal{V}$ in TM is equal to $n+1$.
- For every $m \in M$ the Cauchy characteristic space $C(\mathcal{V})_m$ of $\mathcal{V}$ at m is equal to zero.
- For every $m \in M$ the derived bundle $\mathcal{V}'_m$ has codimension one in $T_m M$.
- For every $m \in M$, $C(\mathcal{V}')_m$ is contained in $\mathcal{V}_m$ and has codimension equal to $2n+1$ in $T_m M$. ⊖

Theorem 4.1.2. *Let $(M, \mathcal{V})$ be a Vessiot system in n variables. Then locally the system is isomorphic to the system defined by a smooth hypersurface in the second order contact bundle for a manifold of dimension $n+1$.*

Proof. This theorem is Proposition 3.8 in Duistermaat [24]. The theorem for the case $n = 2$ is given in Stormark [64, Theorem 11.1]. The original for $n = 2$ is due to Vessiot [68].

Locally the characteristics $C(\mathcal{V}')$ define a projection $\pi : M \to P = M/C(\mathcal{V}')$. The bundle $\mathcal{V}'$ projects to a codimension one bundle $\mathcal{C} \subset TP$. The pair $(P, \mathcal{C})$ is a contact manifold of dimension $2n+1$. For every $m \in M$ the subspace $\mathcal{V}_m$ projects to a codimension $n+1$ (dimension n) subspace $\mathcal{W}_m \subset \mathcal{C}_p$, $p = \pi(m)$. The subspace $\mathcal{W}_m$ defines a point in the second order contact bundle Q over P. The fibers of $Q \to P$ have dimension $n(n+1)/2$. If we vary m along the fiber $\pi^{-1}(p)$ then $\mathcal{W}_m$ will vary in $\mathcal{C}_p$. Let $\iota : M \to Q : m \mapsto \mathcal{W}_m \in Q$.

We will prove that ι is an immersion and hence locally ι defines an embedding $M \to Q$. Let π_Q be the projection $Q \to M$. Since $\pi_Q \circ \iota = \pi$, the kernel of $T\iota$ is contained in $\ker(T\pi) = C(\mathcal{V}')$. Suppose that $X_m \in \ker T_m\iota \subset C(\mathcal{V}')$. Extend X to a smooth vector field contained in $C(\mathcal{V}')$. Since $T_m\iota(\mathcal{V}_m) = \mathcal{W}_m$ and $T_m\iota(X_m) = 0$, it follows that $[X, Y] \subset \mathcal{V} + C(\mathcal{V}') = \mathcal{V}$ for all $Y \subset \mathcal{V}$. But then $X \subset C(\mathcal{V})$. Since $C(\mathcal{V}) = 0$ this implies $X = 0$.

The image of M_p under ι defines a hypersurface in the fiber Q_p over p. Hence M is mapped to a hypersurface in the second order contact bundle Q that is transversal to the projection $Q \to P$. □

The Vessiot theorem says that locally the study of Vessiot systems $(M, \mathcal{V})$ is equivalent to the study of partial differential equations. The transformations that leave invariant the distribution $\mathcal{V}$ are the *contact transformations* of $(M, \mathcal{V})$. In the sections below we will study the geometry of Vessiot systems $(M, \mathcal{V})$.

4.2 Structure on $\mathcal{V}$

We continue our analysis of second order equations. We define a *generalized second order equation* to be a Vessiot system $(M, \mathcal{V})$ in 2 variables. Since locally the generalized second order equations are equivalent to second order scalar partial differential equations by the Vessiot theorem, we will often omit the adjective generalized.

We will use the contact distribution $\mathcal{V}$ and the Lie brackets modulo the subbundle to create more structure on the equation manifold. We specialize to the case $n = 2$, hence $\operatorname{rank} \mathcal{V} = 4$, $\operatorname{rank} \mathcal{V}' = 6$ and $\operatorname{rank} \mathcal{V}'' = \operatorname{rank}(TM) = 7$. The Lie brackets restrict to a bilinear form

$\lambda = [\cdot,\cdot]_{/\mathcal{V}} : \mathcal{V} \times_M \mathcal{V} \to \mathcal{V}'/\mathcal{V}$ on $\mathcal{V}$ to the derived bundle $\mathcal{V}'/\mathcal{V}$ which is called the Lie brackets modulo the subbundle. For any form $\xi \in (\mathcal{V}'/\mathcal{V})^*$ we can make the composition $\xi \circ \lambda$ which is a 2-form on $\mathcal{V}$. Then $(\xi \circ \lambda) \wedge (\xi \circ \lambda)$ defines a 4-form on $\mathcal{V}$. After a choice of a volume form $\mathrm{vol}_{\mathcal{V}}$ on $\mathcal{V}$ we can define a quadratic form Q on $(\mathcal{V}'/\mathcal{V})^*$ by

$$(\xi \circ \lambda) \wedge (\xi \circ \lambda) = Q(\xi)\mathrm{vol}_{\mathcal{V}}. \tag{4.1}$$

The form Q modulo non-zero scalar factors is an invariantly defined conformal quadratic form. The definiteness of this quadratic form is contact invariant and determines the type of the equation. We say the system is *elliptic*, *hyperbolic* or *parabolic* in the case that Q is definite, indefinite or degenerate, respectively. Note that the positive definite case is identified with the negative definite case by means of multiplication of the conformal quadratic form by a factor -1. In the elliptic and hyperbolic case the map $\lambda : \mathcal{V} \times_M \mathcal{V} \to \mathcal{V}'/\mathcal{V}$ is non-degenerate in the sense of Appendix A.1.

Our definition of the type of a Vessiot system agrees with the classical definition of the type of a second order equation, see the example below. Since our definition is contact invariant, this shows immediately that the classical definition is contact invariant as well. A contact invariant definition of the type of a second order scalar equations was also given by Gardner and Kamran [38, p. 63]. Their definition is equivalent to our definition. They make the remark that their definition is contact invariant, but do not compare the definition to the classical definition in terms of the symbol of a partial differential equation.

Example 4.2.1 (Symbol of a second order partial differential equation). In classical theory a second order partial differential equation is a hypersurface in the second order contact bundle of a 3-dimensional manifold that is transversal to the projection to the first order contact manifold. If x, y are two independent coordinates and z the dependent coordinate, then the equation is given by

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0. \tag{4.2}$$

The type of the equation is defined using the symbol of the equation. At a point we can linearize the equation with respect to the highest order variables. The result is a linear partial differential operator and we define the symbol of the equation at the point m to be the symbol of the linearization. The linearization of (4.2) at a point m is given by

$$Lz = F_r(m)\frac{\partial^2 z}{\partial x^2} + F_s(m)\frac{\partial^2 z}{\partial x \partial y} + F_t(m)\frac{\partial^2 z}{\partial y^2}. \tag{4.3}$$

We write X for the manifold with coordinates x, y. The symbol is given by the quadratic form on T^*X

$$(\xi_x, \xi_y) \in T^*X \mapsto F_r{\xi_x}^2 + F_s\xi_x\xi_y + F_t{\xi_y}^2. \tag{4.4}$$

The type of the equation is elliptic or hyperbolic if this quadratic form is definite or indefinite, respectively. We will show that the classical definition of the type of the equation (4.2) corresponds to the type of the Vessiot system defined by this equation.

The second order contact bundle has local coordinates x, y, z, p, q, r, s, t. The contact forms are

$$\begin{aligned}\theta^0 &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y,\\ \theta^1 &= \mathrm{d}p - r\mathrm{d}x - s\mathrm{d}y,\\ \theta^2 &= \mathrm{d}q - s\mathrm{d}x - t\mathrm{d}y.\end{aligned} \tag{4.5}$$

The equation (4.2) defines a hypersurface M in the second order contact bundle. The contact forms θ^j pull back to contact forms on M. The contact distribution $\mathcal{V}$ on M is equal to $\mathrm{span}(\theta^0, \theta^1, \theta^2)^\perp$. We choose $e_5 = \partial_p$, $e_6 = \partial_q$. The vector fields e_5, e_6 define a basis for $\mathcal{V}'/\mathcal{V}$. Define $\xi : \mathcal{V}'/\mathcal{V} \to \mathbb{R}$ by $\xi_1 e^5 + \xi_2 e^6$. So $\xi(a\partial_p + b\partial_q) = a\xi_1 + b\xi_2$.

Recall that $\lambda : \mathcal{V} \times_M \mathcal{V} \to \mathcal{V}'/\mathcal{V}$ was the Lie brackets modulo the subbundle. The composition $\xi \circ \lambda : \mathcal{V} \times_M \mathcal{V} \to \mathbb{R}$ is equal to the restriction of the 2-form

$$\xi_1 \mathrm{d}\theta^1 + \xi_2 \mathrm{d}\theta^2 = \xi_1(\mathrm{d}x \wedge \mathrm{d}r + \mathrm{d}y \wedge \mathrm{d}s) + \xi_2(\mathrm{d}x \wedge \mathrm{d}s + \mathrm{d}y \wedge \mathrm{d}t)$$

to $\mathcal{V} \times_M \mathcal{V}$ and $(\xi \circ \lambda) \wedge (\xi \circ \lambda)/2$ equals the restriction of the 4-form

$$(\xi_1)^2 \mathrm{d}r \wedge \mathrm{d}x \wedge \mathrm{d}s \wedge \mathrm{d}y + (\xi_1\xi_2)\mathrm{d}r \wedge \mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}y + (\xi_2)^2 \mathrm{d}s \wedge \mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}y.$$

We assume that $F_r \neq 0$. If $F_r = 0$, then either $F_s \neq 0$ or $F_t \neq 0$ and we can carry out a calculation similar to the one below leading to the same conclusions. The form $\Omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}s \wedge \mathrm{d}t$ restricts to a non-zero volume form on $\mathcal{V}$ and on $\mathcal{V}$ the contact forms (4.5) vanish. Since the hypersurface M is defined by $F = 0$, the form $F_r\mathrm{d}r + F_s\mathrm{d}s + F_t\mathrm{d}t + F_p\mathrm{d}p + F_q\mathrm{d}q + F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z$ is zero on M. We can use this to calculate that restricted to $\mathcal{V} \times_M \mathcal{V}$

$$F_r(\xi \circ \lambda) \wedge (\xi \circ \lambda) = \left(-F_t(\xi_1)^2 + \xi_1\xi_2 F_s - F_r(\xi_2)^2\right)\Omega.$$

Since $F_r \neq 0$ the conformal quadratic form defining the type of the Vessiot system is

$$\xi \mapsto -F_t(\xi_1)^2 + \xi_1\xi_2 F_s - F_r(\xi_2)^2.$$

If we compare this form to the conformal quadratic form (4.4) then we see that the discriminant of both quadratic forms is $(F_s)^2 - 4F_tF_r$. Hence the two definitions of the type of a second order equation agree. $\oslash$

If Q is non-degenerate then the isotropic cone (see page 38) consists of two distinct lines. We can choose two non-zero points ζ_1, ζ_2 such that each point is contained in a different line of the isotropic cone. In the elliptic case these points are complex. The isotropic elements ζ_1 and ζ_2 are forms on $\mathcal{V}'/\mathcal{V}$ and we can define the characteristic 2-forms

$$\zeta_{1,2} \circ \lambda : \mathcal{V} \times_M \mathcal{V} \to \mathbb{R}.$$

(in the elliptic case the quadratic forms are complex valued). Since $(\zeta_j \circ \lambda) \wedge (\zeta_j \circ \lambda) = 0$ the characteristic 2-forms are decomposable. We can then define the *characteristic distributions*

$\mathcal{F} = \ker(\zeta_1 \circ \lambda)$ and $\mathcal{G} = \ker(\zeta_2 \circ \lambda)$. The distributions $\mathcal{F}$ and $\mathcal{G}$ are called *Monge systems* in the classical literature (see Stormark [64]). For $X \subset \mathcal{F}$, $Y \subset G$ and $Z \subset \mathcal{V}$ we have

$$(\zeta_1 \circ \lambda)(X, Z) = 0, \quad (\zeta_2 \circ \lambda)(Z, Y) = 0.$$

This implies that $(\zeta_1 \circ \lambda)(X, Y) = (\zeta_2 \circ \lambda)(X, Y) = 0$. But then $(\zeta \circ \lambda)(X, Y) = 0$ for all $\zeta \in (\mathcal{V}'/\mathcal{V})^*$ and therefore $\lambda(X, Y) = 0$. Classically this property of the Monge systems is written as $[\mathcal{F}, \mathcal{G}] \equiv 0 \mod \mathcal{V}$.

Example 4.2.2 (Wave equation). The second order scalar equation $z_{xy} = 0$ is called the *wave equation*. We use the classical coordinates x, y, z, p, q, r, t as coordinates on the equation manifold. The Monge systems for the wave equation are given by

$$\mathcal{F} = \operatorname{span}\left(\partial_x + p\partial_z + r\partial_p, \partial_r\right), \quad \mathcal{G} = \operatorname{span}\left(\partial_y + q\partial_z + r\partial_q, \partial_t\right).$$

The derived sequence of $\mathcal{F}$ is given by

$$\mathcal{F}' = \operatorname{span}\left(\partial_x + p\partial_z, \partial_r, \partial_p\right), \quad \mathcal{F}'' = \operatorname{span}\left(\partial_x, \partial_r, \partial_p, \partial_z\right)$$

and $\mathcal{F}''' = \mathcal{F}''$. The distribution $\mathcal{F}$ has three invariants: y, q and t. ⊘

Example 4.2.3. We consider the hyperbolic second order partial differential equation $r + s = 0$. We use the variables x, y, z, p, q, r, t as coordinates on the equation manifold. The contact distribution $\mathcal{V}$ is given by $\operatorname{span}(D_x, D_y, \partial_r, \partial_t)$ with $D_x = \partial_x + p\partial_z + r\partial_p - r\partial_q$, $D_y = \partial_y + q\partial_z - r\partial_p + t\partial_q$. The derived bundle is $\mathcal{V}' = \operatorname{span}(\mathcal{V}', \partial_p, \partial_q)$. We define $e_5 = \partial_p$, $e_6 = \partial_q$. The projections of e_5 and e_6 in $\mathcal{V}'/\mathcal{V}$ we denote by $\tilde{e}_5$ and $\tilde{e}_6$, respectively.

The equation is already linear in the highest order variables. Hence at every point the symbol of the partial differential equation is given by

$$\xi \mapsto (\xi_x)^2 + \xi_x \xi_y. \tag{4.6}$$

With respect to the basis $e_1 = D_x$, $e_2 = D_y$, $e_3 = \partial_r$, $e_4 = \partial_t$ for $\mathcal{V}$ introduced above, the Lie brackets modulo the subbundle are given by the matrix

$$\begin{pmatrix} 0 & 0 & -\tilde{e}_5 + \tilde{e}_6 & 0 \\ 0 & 0 & \tilde{e}_5 & -\tilde{e}_6 \\ \tilde{e}_5 - \tilde{e}_6 & -\tilde{e}_5 & 0 & 0 \\ 0 & \tilde{e}_6 & 0 & 0 \end{pmatrix}.$$

Let $\xi \in (\mathcal{V}'/\mathcal{V})^*$ be given by $\xi_1 \tilde{e}^5 + \xi_2 \tilde{e}^6$. Then $\xi \circ \lambda$ has the matrix representation

$$\begin{pmatrix} 0 & 0 & -\xi_1 + \xi_2 & 0 \\ 0 & 0 & \xi_1 & -\xi_2 \\ \xi_1 - \xi_2 & -\xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 \end{pmatrix}.$$

As a two-form restricted to $\mathcal{V} \times_M \mathcal{V}$ we can represent $\xi \circ \lambda$ as

$$(-\xi_1 + \xi_2)\mathrm{d}x \wedge \mathrm{d}r + \xi_1 \mathrm{d}y \wedge \mathrm{d}r - \xi_2 \mathrm{d}y \wedge \mathrm{d}t.$$

Then $(\xi \circ \lambda) \wedge (\xi \circ \lambda) \in \Lambda^4(\mathcal{V}^*)$ is equal to $2(-\xi_1 + \xi_2)\xi_2\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}r \wedge \mathrm{d}t$. The conformal quadratic form defining the type of the equation is $(\xi_1 - \xi_2)\xi_2$. This quadratic form and the quadratic form in (4.6) both have positive discriminant so the equation is indeed hyperbolic. As isotropic elements we can choose $\zeta_1 = (1, 0)$, $\zeta_2 = (1, 1)$. The characteristic 2-forms on $\mathcal{V} \times_M \mathcal{V}$ are

$$\zeta_1 \circ \lambda = (\mathrm{d}x - \mathrm{d}y) \wedge \mathrm{d}r, \quad \zeta_2 \circ \lambda = \mathrm{d}y \wedge (\mathrm{d}r - \mathrm{d}t).$$

The Monge systems are

$$\mathcal{F} = \operatorname{span}\left(D_x + D_y, \partial_t\right), \quad \mathcal{G} = \operatorname{span}\left(D_x, \partial_r + \partial_t\right).$$

We leave it to the reader to check that the Monge systems satisfy $[\mathcal{F}, \mathcal{G}] \equiv 0 \mod \mathcal{V}$. ⊘

For hyperbolic systems we can use the Monge systems to define an operator J on $\mathcal{V}$ by $J|_{\mathcal{F}} = \mathrm{id}$, $J|_{\mathcal{G}} = -\mathrm{id}$. Note that $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$ so J defines a hyperbolic structure on $\mathcal{V}$. The operator J is invariantly defined up to a minus sign, i.e., up to a choice of one of the two characteristic systems. Since $[\mathcal{F}, \mathcal{G}] \subset \mathcal{V}$ and the Lie brackets modulo the subbundle are non-degenerate, the derived bundle of $\mathcal{F}$ has rank 3. The same is true for $\mathcal{G}$. It follows that $\mathcal{V}' = \mathcal{F}' \oplus \mathcal{G}'$ and $\mathcal{V}'/\mathcal{V} = (\mathcal{F}'/\mathcal{F}) \oplus (\mathcal{G}'/\mathcal{G})$. There is a unique hyperbolic structure $J_{\mathcal{V}'/\mathcal{V}}$ on $\mathcal{V}'/\mathcal{V}$ such that the Lie brackets modulo the subbundle define a map $\mathcal{V} \times_M \mathcal{V} \to \mathcal{V}'/\mathcal{V}$ that is linear with respect to the hyperbolic structures J and $J_{\mathcal{V}'/\mathcal{V}}$. This hyperbolic structure is defined by $J_{\mathcal{V}'/\mathcal{V}}|_{\mathcal{F}'/\mathcal{F}} = \mathrm{id}$, $J_{\mathcal{V}'/\mathcal{V}}|_{\mathcal{G}'/\mathcal{G}} = -\mathrm{id}$. An alternative definition would be $J_{\mathcal{V}'/\mathcal{V}}|_{\ker \zeta_1} = \mathrm{id}$, $J_{\mathcal{V}'/\mathcal{V}}|_{\ker \zeta_2} = -\mathrm{id}$. The hyperbolic structure J on $\mathcal{V}$ can be extended to a hyperbolic structure J on $\mathcal{V}'$ by $J|_{\mathcal{F}'} = \mathrm{id}$, $J|_{\mathcal{G}'} = -\mathrm{id}$. Note that $J^2 = \mathrm{id}$ on $\mathcal{V}'$. Since $\mathcal{V}$ is invariant under J, the extension $J : \mathcal{V}' \to \mathcal{V}'$ induces a map $J : \mathcal{V}'/\mathcal{V} \to \mathcal{V}'/\mathcal{V}$. This map is equal to $J_{\mathcal{V}'/\mathcal{V}}$. Because of the definition of J we will write $\mathcal{V}_+ = \mathcal{F}$ and $\mathcal{V}_- = \mathcal{G}$ for the Monge systems.

In the elliptic case we have complex Monge systems and we can define J as an operator on $\mathcal{V} \otimes \mathbb{C}$. We define $J|_{\mathcal{F}} = i$ and $J|_{\mathcal{G}} = -i$. Since $J^2 = -1$ on $\mathcal{V} \otimes \mathbb{C}$ this indeed defines a complex structure on $\mathcal{V} \otimes \mathbb{C}$. Let us proof that the restriction of J to $\mathcal{V}$ defines a complex structure on $\mathcal{V}$. We can write any vector $X \in \mathcal{V} \otimes \mathbb{C}$ as $X = X_+ + X_-$, with X_+ and X_- the components of X in the characteristic systems $\mathcal{F}$ and $\mathcal{G}$, respectively. The isotropic elements ζ_1 are ζ_2 can be chosen to be complex conjugated. This implies that $(\zeta_1 \circ \lambda) \wedge (\zeta_1 \circ \lambda)$ and $(\zeta_2 \circ \lambda) \wedge (\zeta_2 \circ \lambda)$ are complex conjugated as well and hence $\mathcal{F}$ and $\mathcal{G}$ are complex conjugated. If $X \in \mathcal{V}$, i.e., X is real, then $X = \bar{X}$ and hence $X_+ + X_- = \overline{X_+} + \overline{X_-}$. It follows that $\overline{X_+} = X_-$ and $X = X_+ + \overline{X_+}$. Then $JX = iX_+ - iX_- = iX_+ - i\overline{X_+}$ and hence $\overline{JX} = JX$. So JX is in $\mathcal{V}$ again and J defines an operator $J : \mathcal{V} \to \mathcal{V}$ with $J^2 X = -X$. Again there is a unique extension of J to $\mathcal{V}'$ such that J defines a complex structure on $\mathcal{V}'$ and λ is complex-bilinear.

Remark 4.2.4. In the analytic setting we can complexify any elliptic system and arrive in this way at a hyperbolic system for complex variables. ⊘

4.3 The integral elements

The structure of the integral elements at a point $x \in M$ is completely determined by the type of the equation. We will give a geometric picture of the space of 2-dimensional integral elements $V_2(\mathcal{V})_m$ depending on the type of the equation. Recall from Section 1.2.4 that a 2-plane E is an integral element for the distribution $\mathcal{V}$ if and only if $E \subset \mathcal{V}$ and the Lie brackets modulo $\mathcal{V}$ vanish on E. The 1-dimensional integral elements are the 1-dimensional linear subspaces of $\mathcal{V}$. The distribution $\mathcal{V}$ has no 3-dimensional integral elements.

Proposition 4.3.1. *Let $(M, \mathcal{V})$ be an elliptic or hyperbolic first order system. A linear subspace E of $\mathcal{V}$ of dimension two is an integral element if and only if $J(E) = E$, i.e., E is J-invariant.*

The proof of the proposition follows from the definition of J using the Monge systems and the fact that a linear subspace E of a distribution is an integral element if and only if $E \subset \mathcal{V}$ and the Lie brackets modulo the subbundle vanish when restricted to $E \times E$. Proposition 4.3.1 leads to the following geometric description of the space of 2-dimensional integral elements.

Elliptic case The complex Monge systems define a unique complex structure on $\mathcal{V}$. The integral elements are precisely the complex 1-dimensional complex-linear subspaces in $\mathcal{V}$. So $V_2(\mathcal{V})_m$ is the complex projective line for the complex vector space $(\mathcal{V}_m, J_m)$ and therefore can be identified with the Riemann sphere.

Hyperbolic case Every integral element E has 1-dimensional intersections $E_+ = E \cap \mathcal{F}$, $E_- = E \cap \mathcal{G}$ with the Monge systems. The mapping $(X, Y) \mapsto X + Y$ from $\mathbb{P}\mathcal{F} \times_M \mathbb{P}\mathcal{G}$ to $V_2(\mathcal{V})$ is bijective. Because $\mathbb{P}\mathcal{F}_m$ and $\mathbb{P}\mathcal{G}_m$ are both diffeomorphic to a circle, $V_2(\mathcal{V})_m$ is diffeomorphic to a torus.

Parabolic case For a parabolic equation the two Monge systems coincide and hence there is only one (rank two) characteristic system $\mathcal{F}$. A 2-dimensional subspace of $\mathcal{V}$ is an integral element if and only if the intersection of E and $\mathcal{F}$ is non-zero. The space of integral elements has a cone-type singularity at the integral element $E = \mathcal{F}$ and looks like a constricted torus (see Figure 4.1). The constricted point is equal to the integral element E.

Integral manifolds. Using the Cartan-Kähler theorem one can prove in the analytic setting that for each integral element E at x there exists an integral manifold through the point x with tangent space equal to E. These integral manifolds can be parameterized using two functions of one variable. See Theorem 6.1.6.

The 2-dimensional integral manifolds of $\mathcal{V}$ have at each point a tangent space that is equal to a 2-dimensional integral element E. The 2-plane E is J-invariant and hence each element E has a complex structure or a hyperbolic structure depending on the type of the equation. On the integral manifold this defines an almost complex structure or almost product structure depending on the type of equation. Since the dimension is two this structure is always integrable. This means that the 2-dimensional integral manifolds of a hyperbolic equation have a double foliation by characteristic curves (see Example 2.2.9).

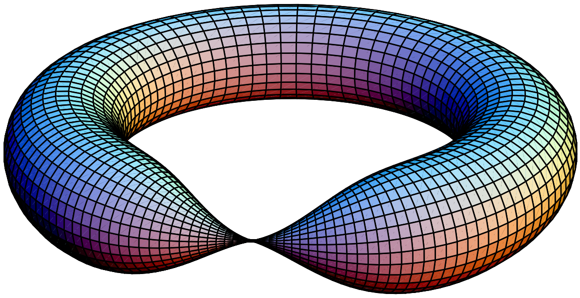

Figure 4.1: Space of 2-dimensional integral elements for the parabolic equations

4.4 The Nijenhuis tensor

Until this point the structure for elliptic, hyperbolic and parabolic cases were very similar and we could use the same constructions. From this point on we will assume that the system is either elliptic or hyperbolic. These two cases are very similar and although the proofs and constructions in both cases are not identical, they closely related.

In the elliptic case we have constructed a complex structure $J_{\mathcal{V}}$ on $\mathcal{V}$ and a complex structure $J_{\mathcal{V}'/\mathcal{V}}$ on $\mathcal{V}'$. In the hyperbolic case we have a hyperbolic structure on $\mathcal{V}$ and $\mathcal{V}'/\mathcal{V}$. We have given an extension of the complex structure to $\mathcal{V}'$ using the derived bundles of the Monge systems. To give additional motivation for this extension of J we will consider the Nijenhuis tensor. For convenience the analysis below will be done for the elliptic case. However, in the hyperbolic case everything will be the same with complex replaced by hyperbolic.

Recall that the Nijenhuis tensor for an almost complex structure or almost product structure J is given by the expression

$$N(X, Y) = [J, J](X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]. \tag{4.7}$$

Since J is only defined on $\mathcal{V}'$ this expression is not well-defined for all vectors X, Y. But for $X, Y \subset \mathcal{V}$ we have $JX, JY, [X, Y], [JX, Y], [X, JY] \subset \mathcal{V}'$ so we can define the map

$$N : \mathcal{V} \times_M \mathcal{V} \to TM.$$

We will call this map the Nijenhuis tensor as well. We will consider extensions of the complex structure on $\mathcal{V}$ to a complex structure on $\mathcal{V}'$ such that the induced structure on $\mathcal{V}'/\mathcal{V}$ makes the Lie brackets modulo the subbundle complex-linear. We will see that there is a unique extension of the complex structure to $\mathcal{V}'$ such that the Nijenhuis tensor vanishes on $\mathcal{V} \times_M \mathcal{V}$.

By choosing a splitting s of the exact sequence

$$0 \to \mathcal{V} \to \mathcal{V}' \overset{s}{\leftrightarrows} \mathcal{V}'/\mathcal{V} \to 0$$

we can identify $\mathcal{V}'$ with $\mathcal{V}\times_M\mathcal{V}'/\mathcal{V}$. Any such identification will extend our complex structure to a complex structure $J : \mathcal{V}' \to \mathcal{V}'$ such that

$$J|_{\mathcal{V}} = J_{\mathcal{V}}, \quad J|_{\mathcal{V}'/\mathcal{V}} = J_{\mathcal{V}'/\mathcal{V}}.$$

The complex structure J has the property $J(\mathcal{V}) \subset \mathcal{V}$. Let K be another complex structure on $\mathcal{V}'$ for which the restriction to $\mathcal{V}$ and $\mathcal{V}'/\mathcal{V}$ equals $J_{\mathcal{V}}$ and $J_{\mathcal{V}'/\mathcal{V}}$, respectively. Write $D = K - J$. Then for $X \in \mathcal{V}$ we have

$$D(X) = K(X) - J(X) = J_{\mathcal{V}}(X) - J_{\mathcal{V}}(X) = 0.$$

Hence $D(\mathcal{V}) = 0$. Since $D(\mathcal{V}) = 0$ and D is a map from $\mathcal{V}'$ to $\mathcal{V}'$ it induces a map $D|_{\mathcal{V}'/\mathcal{V}}$: $\mathcal{V}'/\mathcal{V} \to \mathcal{V}'/\mathcal{V}$. But $D|_{\mathcal{V}'/\mathcal{V}} = K_{\mathcal{V}'/\mathcal{V}} - J_{\mathcal{V}'/\mathcal{V}} = 0$, so $D(\mathcal{V}') \subset \mathcal{V}$. This implies $D^2 = 0$, hence D acts as a boundary operator. Finally we have

$$\begin{aligned} 0 &= K^2 - J^2 = (J + D)^2 - J^2 \\ &= J^2 + JD + DJ + D^2 - J^2 = JD + DJ. \end{aligned} \tag{4.8}$$

Hence D is complex-antilinear with respect to the complex structure J (and complex-antilinear with respect to the structure K as well). The complex-antilinear maps $D : \mathcal{V}' \to \mathcal{V}'$ with the properties

$$D(\mathcal{V}) = 0, \quad D(\mathcal{V}') \subset \mathcal{V}, \tag{4.9}$$

can be described in the following way. Choose a complex-antilinear form $\upsilon : \mathcal{V}'/\mathcal{V} \to \mathbb{C}$ with $\upsilon \neq 0$; such a form is unique up to a complex scalar factor. We can regard υ as a complex-antilinear form on $\mathcal{V}'$ that is identically zero on $\mathcal{V}$. The map

$$\delta \mapsto \big(D_\delta : \mathcal{V}' \to \mathcal{V}' : X \mapsto \upsilon(X)\delta\big)$$

from $\mathcal{V}$ to the space of complex-antilinear maps $D : \mathcal{V}' \to \mathcal{V}'$ with the properties (4.9) is bijective.

To make our extension to J to $\mathcal{V}'$ unique we will consider the Nijenhuis tensor. It is clear that N restricted to $\mathcal{V} \times_M \mathcal{V}$ takes values in $\mathcal{V}'$. From the J-linearity of the Lie brackets modulo $\mathcal{V}$ it follows that in fact $N(\mathcal{V}, \mathcal{V}) \subset \mathcal{V}$. So the Nijenhuis tensor defines an anti-symmetric bi-J-antilinear $\mathcal{V}$-valued form on $\mathcal{V}$. By counting dimensions we can easily see that such a form is uniquely determined by the value on a pair $X, Y \in \mathcal{V}$. We can write $N(X, Y) = w(X, Y)\nu$ for $\nu \in \mathcal{V}$ and w an anti-symmetric bi-antilinear $\mathbb{C}$-valued form (or $\mathbb{D}$-valued form in the hyperbolic setting).

Let $K = J + D$ as above and assume $D = D_\delta$ for a certain $\delta \in \mathcal{V}$. Then for $X, Y \subset \mathcal{V}$

$$\begin{aligned} [K, K](X, Y) &= [(J + D)X, (J + D)Y] - (J + D)[(J + D)X, Y] \\ &\qquad - (J + D)[X, (J + D)Y] + (J + D)^2[X, Y] \\ &= [JX, JY] - (J + D)[JX, Y] - (J + D)[X, JY] + J^2[X, Y] \\ &= [J, J](X, Y) - D[JX, Y] - D[X, JY] \\ &= w(X, Y)\nu - \upsilon([JX, Y] + [X, JY])\delta \\ &= w(X, Y)\nu - \upsilon(2J[X, Y])\delta. \end{aligned}$$

The map $(X, Y) \mapsto \upsilon(2J[X, Y])$ is an anti-symmetric bi J-antilinear $\mathbb{C}$-valued form, just as w. Such a form is determined up to a scalar factor. Hence there is a unique $\delta \in \mathcal{V}$ such that $w(X, Y)\nu - \upsilon(2J[X, Y])\delta = 0$ and hence the Nijenhuis tensor $[K, K]$ for K vanishes on $\mathcal{V}$. We have proved

Proposition 4.4.1. *For an elliptic (hyperbolic) system there is a unique almost complex structure (almost product structure) J on $\mathcal{V}'$ such that the Lie brackets modulo $\mathcal{V}$ are J-linear and the Nijenhuis tensor $N = [J, J]$ vanishes on $\mathcal{V} \times_M \mathcal{V}$.*

Finally we will show that the almost complex structure defined in the proposition above using the Nijenhuis tensor and almost complex structure defined in the previous section using the derived bundles of the Monge systems are identical. Let J be the almost complex structure defined by the Monge systems. Assume that X is a vector field in $\mathcal{F}$ and Y a vector field in G. Then $JX = iX$, $JY = -iY$ and therefore

$$\begin{aligned} N(X, Y) &= [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] \\ &= [iX, -iY] - J[iX, Y] - J[X, -iY] - [X, Y] \\ &= [X, Y] - iJ[X, Y] + iJ[X, Y] - [X, Y] = 0. \end{aligned}$$

In a similar way we can prove that for $X, Y \in \mathcal{F}$ and $X, Y \in \mathcal{G}$ we have $N(X, Y) = 0$. This shows that the Nijenhuis tensor vanishes on $\mathcal{V} \times_M \mathcal{V}$ for this almost complex structure and hence must be equal to the almost complex structure from Proposition 4.4.1.

There is no complex structure or almost product structure on the equation manifold since the equation manifold is of dimension 7. The operator J is only defined as a map $\mathcal{V}' \to \mathcal{V}'$ and therefore we cannot define N on the whole tangent space. Can we extend J or N in a natural way to a larger domain?

Let us start with an analysis of the Nijenhuis tensor. We could define the Nijenhuis tensor on $\mathcal{V} \times_M \mathcal{V}$ because for all $X, Y \subset \mathcal{V}$ we have $[X, Y] \subset \mathcal{V}'$ and J is defined on $\mathcal{V}'$. We can in fact define $N(X, Y)$ for all vector fields $X, Y \subset \mathcal{V}'$ for which $[X, Y] \subset \mathcal{V}'$. Note that this condition only depends on the values of X and Y at a point and not on the first order derivatives of X and Y. The Lie brackets modulo $\mathcal{V}'$ define an antisymmetric bilinear map $\kappa : \mathcal{V}' \times_M \mathcal{V}' \to TM/\mathcal{V}'$. The set $\Gamma = \{ (X, Y) \in \mathcal{V}' \times_M \mathcal{V}' \mid \kappa(X, Y) = 0 \}$ is the largest set on which we can define the Nijenhuis tensor using the complex structure $J : \mathcal{V}' \to \mathcal{V}'$. The structure of the set Γ_m can vary with the point $m \in M$ and also depends on the system $(M, \mathcal{V})$. For this reason we will restrict ourselves to a subset of Γ.

The condition $[X, Y] \subset \mathcal{V}'$ for all $Y \subset \mathcal{V}'$ is precisely the definition of the Cauchy characteristic subspace $C(\mathcal{V}')$. We can therefore define

$$N : \mathcal{V}' \times_M C(\mathcal{V}') \to \mathcal{V}' : (X, Y) \mapsto [J, J](X, Y).$$

This gives an extension of the tensor N defined in the previous section at each point $m \in M$ to the set $(\mathcal{V}_m \times \mathcal{V}_m) \cup (\mathcal{V}'_m \times C(\mathcal{V}')_m)$. If we use the Nijenhuis tensor in the context of second order partial differential equations we will mean this extension, unless stated otherwise. We will use this extension later to characterize the Monge-Ampère equations. Since N is identically zero when restricted to $\mathcal{V} \times_M \mathcal{V}$ the interesting properties of the Nijenhuis tensor are determined by the restriction to $\mathcal{V}' \times_M C(\mathcal{V}')$. The extension satisfies the property $N|_{\mathcal{V}\times_M C(\mathcal{V}')} = 0$ and therefore is completely determined by the restriction of N to $(\mathcal{V}'/\mathcal{V}) \times_M C(\mathcal{V}')$. Both $\mathcal{V}'/\mathcal{V}$ and $C(\mathcal{V}')$ are complex one-dimensional. This together with the complex-antilinearity of N implies that the tensor N is completely determined by its value on a pair of non-zero vectors $X \in \mathcal{V}'/\mathcal{V}$, $Y \in C(\mathcal{V}')$. For hyperbolic equations the Nijenhuis tensor is determined by its value on a pair of vectors $X \in \mathcal{V}'/\mathcal{V}$, $Y \in C(\mathcal{V}')$ that are generic with respect to the hyperbolic structure. For any generic element $Y \in C(\mathcal{V}')$ we can define the map

$$A = A(Y) : \mathcal{V}' \to \mathcal{V}' : X \mapsto N(X, Y).$$

The image of A equals the image $\mathcal{D}$ of N. For an elliptic system the image of N can have complex rank 0 or 1. For a hyperbolic system the rank of the image can be 0, 1 or 2. If the image is non-zero, then it is not contained in $\mathcal{V}$. This follows from theory to be developed in Chapter 6. The statement follows from Lemma 6.1.2.

Example 4.4.2. Consider the hyperbolic equations of the form

$$s = \phi(x, y, z, p, q).$$

If the equation is Darboux integrable (see Section 8) with 2 or 3 invariants for each characteristic system, then the equation is called a *hyperbolic Goursat equation*.

We define $D_x = \partial_x + p\partial_z + r\partial_p + \phi\partial_q$, $D_y = \partial_y + q\partial_z + \phi\partial_p + t\partial_q$ and

$$\begin{aligned} F_1 &= D_x + D_y(\phi)\partial_t, \quad F_2 = \partial_r, \\ G_1 &= D_y + D_x(\phi)\partial_r, \quad G_2 = \partial_t, \\ F_3 &= \partial_p, \quad G_3 = \partial_q, \quad Z = \partial_z. \end{aligned}$$

The characteristic systems are given by $\mathcal{F} = \operatorname{span}(F_1, F_2)$, $\mathcal{G} = \operatorname{span}(G_1, G_2)$. The derived bundle of $\mathcal{V}$ is spanned by $\mathcal{F}$, $\mathcal{G}$, F_3 and G_3. The Cauchy characteristics are $C(\mathcal{V}') = \operatorname{span}(F_2, G_2)$. The hyperbolic structure on $\mathcal{V}'$ is given by

$$\begin{aligned} J(F_1) &= F_1, \quad J(F_2) = F_2, \quad J(F_3) = F_3, \\ J(G_1) &= G_1, \quad J(G_2) = G_2, \quad J(G_3) = G_3. \end{aligned}$$

The Nijenhuis tensor on $(\mathcal{V}'/\mathcal{V}) \times_M C(\mathcal{V}')$ is identically zero. $\oslash$

Example 4.4.3 ($s = r^2$). The characteristic systems are given by

$$\mathcal{F} = \operatorname{span}\big(F_1 = D_x, F_2 = \partial_r + 4r^2\partial_t\big), \quad \mathcal{G} = \operatorname{span}\big(G_1 = D_y - 2rD_y, G_2 = \partial_t\big).$$

Let $F_3 = [F_1, F_2] = -\partial_p - 2r\partial_q$, $G_3 = [G_1, G_2] = -\partial_q$. The hyperbolic structure acts as the identity on $\mathcal{F}' = \operatorname{span}(F_1, F_2, F_3)$ and as minus the identity on $\mathcal{G}' = \operatorname{span}(G_1, G_2, G_3)$. The Nijenhuis tensor is determined by

$$N(F_3, F_2) = 0, \quad N(G_3, G_2) = 8\partial_q.$$

The image $\mathcal{D}$ of the Nijenhuis tensor has rank 1. Note that $N(G_3, G_2) \not\equiv 0 \mod \mathcal{V}$. $\oslash$

Example 4.4.4 ($3rt^2 + 1 = 0$). Consider the equation $3rt^3 + 1 = 0$ for $t \neq 0$. The characteristic systems are given by

$$\mathcal{F} = \operatorname{span}\big(F_1 = D_x - (1/t^2)D_y,\, F_2 = \partial_t - (1/t^2)\partial_s\big),$$
$$\mathcal{G} = \operatorname{span}\big(G_1 = D_x + (1/t^2)D_y,\, G_3 = \partial_t + (1/t^2)\partial_s\big),$$

with $D_x = \partial_x + p\partial_z - (1/(3t^3))\partial_p + s\partial_q$ and $D_y = \partial_y + q\partial_z + s\partial_p + t\partial_q$.

The image of the Nijenhuis tensor on $\mathcal{F}$ is spanned by $b_1 = -t\partial_y - tq\partial_q + (1 - st)\partial_p$; the image of N on $\mathcal{G}$ is spanned by $b_2 = t\partial_y + tq\partial_q + (1 + st)\partial_p$. So $\mathcal{D}$ has rank 2. Since $[b_1, b_2] = 0$ the bundle $\mathcal{D}$ is integrable. ⊘

Example 4.4.5 (Generic image for the Nijenhuis tensor). Consider the equation $r = \phi(x, y)t^2$ with $\phi(x, y)$ an arbitrary function with $\phi(x, y) > 0$. For points with $t > 0$ this defines a hyperbolic equation. Let $\kappa = \sqrt{2\phi(x, y)t}$. The Monge systems are given by

$$\mathcal{F} = \operatorname{span}\big(F_1 = D_x + \kappa D_y - tD_y(\kappa)\partial_t,\, F_2 = \partial_s + \kappa^{-1}\partial_t\big),$$
$$\mathcal{G} = \operatorname{span}\big(G_1 = D_x - \kappa D_y + tD_y(\kappa)\partial_t,\, G_2 = \partial_s - \kappa^{-1}\partial_t\big).$$

The image $\mathcal{D}$ of the Nijenhuis tensor has rank 2. The derived bundle $\mathcal{D}'$ has rank 3. ⊘

Example 4.4.6. We consider the second order equation $s = pq$. This is a Monge-Ampère equation that is Darboux integrable on the second order jet bundle. In local coordinates x, y, z, p, q, r, t we define the vector fields

$$F_1 = \partial_x + p\partial_z + r\partial_p + pq\partial_q + (pt + pq^2)\partial_t, \quad F_2 = \partial_r, \quad F_3 = -\partial_p,$$
$$G_1 = \partial_y + q\partial_z + pq\partial_p + t\partial_q + (pr + pq^2)\partial_t, \quad G_2 = \partial_t, \quad G_3 = -\partial_q.$$

The distribution $\mathcal{V}$ is spanned by F_1, G_1, F_2, G_2. The Nijenhuis tensor restricted to $C(\mathcal{V}') \times_M (\mathcal{V}'/\mathcal{V})$ is determined by $N(F_2, F_3)$ and $N(G_2, G_3)$. We can check that

$$N(F_2, F_3) = 0, \quad N(G_2, G_3) = 0.$$

So the Nijenhuis tensor is identically zero. ⊘

4.5 Invariant framings

If we assume the system is generic in the sense that $\operatorname{rank}\mathcal{D} = 2$ and perhaps some other conditions we can adapt to framing even further. Recall that for each non-zero generic $Y \subset C(\mathcal{V}')$ we have defined the map $A(Y) : \mathcal{V}' \to \mathcal{V}'$. We can restrict this map to the image $\mathcal{D}$ of the Nijenhuis tensor. We then get a map $A(Y) : \mathcal{D} \to \mathcal{D}$ that is complex-antilinear or hyperbolic-antilinear in the elliptic and hyperbolic case, respectively. If $\operatorname{rank}\mathcal{D} = 2$, then $\operatorname{rank}(\mathcal{D}/\mathcal{V}) = 2$ as well. This fact can be proved using Lemma 6.1.2. Because $A(\mathcal{V}) = 0$ it follows that the map $A(Y) : \mathcal{D} \to \mathcal{D}$ has rank two and is invertible. The element Y is unique up to a factor ϕ in $\mathbb{C}$ or $\mathbb{D}$. The map $\rho = A(Y) \circ A(Y) : \mathcal{D} \to \mathcal{D}$ is complex-linear or hyperbolic-linear and invertible. This means that the map ρ is determined by scalar factor, i.e., $\rho : \mathcal{D} \to \mathcal{D} : X \mapsto \psi X$. Then

$$A(\phi Y) \circ A(\phi Y) = |\phi|^2 \rho = (|\phi|^2 \psi)\,\mathrm{id}\,.$$

We can choose ϕ such that the norm of $|\phi^2|\rho$ equals 1. This determines an element $\phi Y \in C(\mathcal{V}')$ up to a rotation (a rotation is multiplication by an element $\exp(i\phi)$ in the elliptic case and multiplication by $(a, 1/a)^T$ in the hyperbolic case). We will return to the problem of finding an invariant (co)framing when we have developed the necessary structure theory using differential forms. See Section 6.1.3.

4.6 First order systems

For first order systems of partial differential equations we have a 6-dimensional equation manifold M and a distribution $\mathcal{V}$ of rank 4. The non-degeneracy of the Lie brackets modulo $\mathcal{V}$ implies that the derived bundle has rank 6; it is equal to TM. Just as for second order equations we can analyze the Lie brackets modulo the subbundle. Since for both first order systems and second order equations the distribution $\mathcal{V}$ satisfies $\operatorname{rank} \mathcal{V} = 4$, $\operatorname{rank} \mathcal{V}' = 6$ the analysis of the Lie brackets leads to the same structures. In particular we find a conformal quadratic form on $(\mathcal{V}'/\mathcal{V})^*$ and define the system to be elliptic or hyperbolic if this form is definite or indefinite. Depending on the type of the system we find two Monge systems $\mathcal{V}_+ = \mathcal{F}$, $\mathcal{V}_- = \mathcal{G}$ in $\mathcal{V}$ or $\mathcal{V} \otimes \mathbb{C}$. For hyperbolic systems we define a hyperbolic structure J on $\mathcal{V}$ by $J|_{\mathcal{V}_\pm} = \pm 1$ and for elliptic systems we define a complex structure on $\mathcal{V} \otimes \mathbb{C}$ by $J|_{\mathcal{V}_\pm} = \pm i$. The integral elements have the same structure as for second order equations.

Theorem 4.6.1. *Let $(M, \mathcal{V})$ be a generalized elliptic (hyperbolic) first order system. There is a unique complex structure (almost product structure) J on $TM = \mathcal{V}'$ such that:*

- *The Lie brackets modulo the subbundle define a complex-bilinear (or hyperbolic-bilinear) map $\lambda : \mathcal{V} \times_M \mathcal{V} \to TM$.*
- *The Nijenhuis tensor $N = [J, J]$ is identically zero when restricted to $\mathcal{V} \times_M \mathcal{V}$.*

We define the type of a generalized first order system using the conformal quadratic form on $\mathcal{V}'/\mathcal{V}$ defined by $\xi \mapsto (\xi \circ \lambda) \wedge (\xi \circ \lambda)$. Just as for second order equations (see Example 4.2.1), this definition of type corresponds to the classical definition of the type of a first order system. The symbol for a first order system is a quadratic form on the cotangent space with values in the 2×2-matrices. By taking the determinant of these 2×2-matrices we find a conformal quadratic form that determines the type.

There is another (equivalent) definition of the type of a first order system $M \subset \mathrm{Gr}_2(TB)$. The fibers $M_b \subset \mathrm{Gr}_2(T_bB)$ over a point $b \in B$ are surfaces in the Grassmannian $\mathrm{Gr}_2(T_bB)$. In Section 2.3.1 we defined the type of a surface in the Grassmannian. A first order system is elliptic or hyperbolic precisely when the fibers M_b, $b \in B$ are elliptic or hyperbolic surfaces in the Grassmannian $\mathrm{Gr}_2(T_bB)$, see Remark 4.6.2.

Remark 4.6.2 (Type of first order system). Suppose we have a first order system $M \subset \mathrm{Gr}_2(TB)$. We choose local coordinates x, y, u, v for B. We let x, y, u, v, p, q, r, s be the corresponding local coordinates for $\mathrm{Gr}_2(TB)$. The equation manifold M has codimension two in $\mathrm{Gr}_2(TB)$ and is transversal to the projection $\mathrm{Gr}_2(TB) \to M$. This means that we can give a parameterization of M using two coordinates a, b and

$$\begin{aligned} p &= p(x, y, u, v, a, b), \quad & q &= q(x, y, u, v, a, b), \\ r &= r(x, y, u, v, a, b), \quad & s &= s(x, y, u, v, a, b). \end{aligned}$$

For each point $b \in B$ the 2×2-matrices

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

form local coordinates for $\mathrm{Gr}_2(T_b B)$. The conformal quadratic form on the tangent space is given by the determinant, see Section 2.1.1. The tangent spaces to the fibers M_b are spanned by the two vectors

$$A = \begin{pmatrix} p_a & q_a \\ r_a & s_a \end{pmatrix}, \quad B = \begin{pmatrix} p_b & q_b \\ r_b & s_b \end{pmatrix}. \tag{4.10}$$

A vector $\mu_1 A + \mu_2 B$ in the tangent space of M_b is in the isotropic cone for the conformal quadratic form if and only if

$$\det(\mu_1 A + \mu_2 B) = \det \begin{pmatrix} \mu_1 p_a + \mu_2 p_b & \mu_1 q_a + \mu_2 q_b \\ \mu_1 r_a + \mu_2 r_b & \mu_1 s_a + \mu_2 s_b \end{pmatrix} = 0. \tag{4.11}$$

The expression above is a quadratic form in μ_1, μ_2 and on page 46 we defined the type of the tangent plane using the sign of the discriminant of this quadratic form.

Recall that the type of the system $(M, \mathcal{V})$ was defined using a conformal quadratic form defined in terms of the Lie brackets modulo $\mathcal{V}$. A choice of isotropic elements for this conformal quadratic form leads to the Monge systems (which are complex in the elliptic case). For the projection $M \to B$ the tangent spaces to the fibers are contained in $\mathcal{V}$ and are integral elements of $\mathcal{V}$. The vectors X in $\mathcal{V}$ (or $\mathcal{V} \otimes \mathbb{C}$) that are contained in one of the Monge systems have the property that the rank of the map $\mathcal{V} \to \mathcal{V}'/\mathcal{V} : Z \mapsto \lambda(X, Z)$ is one, instead of two for a generic element X.

In the local coordinates introduces above the contact distribution $\mathcal{V}$ is spanned by

$$\begin{aligned} &X = \partial_x + p\partial_u + r\partial_v, \quad Y = \partial_y + q\partial_u + s\partial_v, \\ &A = \partial_a, \quad B = \partial_b. \end{aligned} \tag{4.12}$$

The tangent space to the fibers is spanned by the vectors A, B. Note that the vector A in TM corresponds to the vector $p_a\partial_p + q_a\partial_q + r_a\partial_r + s_a\partial_s$ in $T\,\mathrm{Gr}_2(TB)$. Hence the vectors A, B in (4.12) correspond to the vectors A, B defined in (4.10), through the embedding $M \to \mathrm{Gr}_2(TB)$. We use ∂_u and ∂_v as representatives for a basis of $\mathcal{V}'/\mathcal{V}$. Let $V = \mu_1 A + \mu_2 B$.

Then

$$\begin{aligned}
\lambda(V, X) &= (\mu_1 p_a + \mu_2 p_b)\partial_u + (\mu_1 r_a + \mu_2 r_b)\partial_v \quad \bmod \mathcal{V},\\
\lambda(V, Y) &= (\mu_1 q_a + \mu_2 q_b)\partial_u + (\mu_1 s_a + \mu_2 s_b)\partial_v \quad \bmod \mathcal{V},\\
\lambda(V, A) &= 0,\\
\lambda(V, B) &= 0.
\end{aligned} \tag{4.13}$$

The rank of the image of the map $Z \mapsto \lambda(V, Z)$ is equal to the rank of the matrix

$$\begin{pmatrix} \mu_1 p_a + \mu_2 p_b & \mu_1 r_a + \mu_2 r_b \\ \mu_1 q_a + \mu_2 q_b & \mu_1 s_a + \mu_2 s_b \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.14}$$

This rank is one if and only if the determinant of the upper 2×2-part of this matrix is zero. This determinant is equal to the determinant in (4.11). This proves that the type of the system is equal to the type of each surface in $\mathrm{Gr}_2(T_b B)$. Moreover, the characteristic lines of the surface M_b in $\mathrm{Gr}_2(T_b B)$ are equal to the intersections of the tangent space to M_b with the Monge systems of the system $(M, \mathcal{V})$. ⊘

4.6.1 Vessiot theorem for first order systems

We will prove in this section that in the analytic setting a generalized first order system $(M, \mathcal{V})$ is locally equivalent to a first order system of two partial differential equations for two functions in two variables. We start with a weaker version of the theorem that is true also in the C^∞ setting.

Theorem 4.6.3 (Weak Vessiot theorem). *Let $(M, \mathcal{V}, \mathcal{U})$ be generalized first order system under point geometry. On M the distribution $\mathcal{U}$ is a rank two integrable subbundle of $\mathcal{V}$.*

Then locally the quotient of M by $\mathcal{U}$ defines a base manifold B and the first order system M is canonically equivalent to a first order system $\tilde{M} \subset \mathrm{Gr}_2(TB)$.

Proof. Let $\pi : M \to B$ be the projection to the quotient of M by the leaves of $\mathcal{U}$. In general this projection is only locally defined. Then π is a smooth submersion and $T\pi$ maps $\mathcal{V}_m$ to a 2-dimensional linear subspace of $T_b M$, $b = \pi(m)$. Now define $\phi : M \to \mathrm{Gr}_2(TB)$ by $m \mapsto T_m\pi(\mathcal{V}_m) \in \mathrm{Gr}_2(T_{\pi(m)}B)$. The fact that the Lie brackets on $\mathcal{V}$ are non-degenerate implies that the map ϕ is an immersion and hence ϕ locally defines a diffeomorphism $M \to \tilde{M} = \phi(M) \subset \mathrm{Gr}_2(TB)$. Let $\tilde{\pi}$ be the projection $\tilde{M} \to B$.

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\pi"] & \tilde{M} \subset \mathrm{Gr}_2(TB) \arrow[dl, "\tilde{\pi}"] \\ B & \end{tikzcd}$$

We have to prove that the distribution $\mathcal{V}$ is mapped to the distribution $\tilde{\mathcal{V}}$ dual to the pull-back of the contact system on $\mathrm{Gr}_2(TB)$. Note that $\pi = \tilde{\pi} \circ \phi$. For a vector $X \in \mathcal{V}_m$ we have $T_m\pi(X) \in \phi(m) \subset T_{\pi(m)}B$. At the same time $T_m\pi(X) = (T_{\phi(m)}\tilde{\pi})(T_m\phi X)$. So the vector $\tilde{X} = T_m\phi(X) \in T_{\phi(m)}\tilde{M} \subset T_{\phi(m)}\,\mathrm{Gr}_2(TB)$ is mapped under $\tilde{\pi}$ to a vector in $\phi(m)$. But then $\tilde{X}$ is in the contact distribution on $\mathrm{Gr}_2(TB)$ (for the definition of this contact distribution see Section 1.2.2). $\square$

To prove that any generalized first order system $(M, \mathcal{V})$ is locally equivalent to a first order system of partial differential equations it is sufficient to show that there exists a local foliation of M by 2-dimensional integral manifolds of $\mathcal{V}$. Then the previous theorem shows the system is equivalent to a first order system of partial differential equations. We will construct a linear Pfaffian system in involution for which the solutions are integral manifolds of $\mathcal{V}$. Then the Cartan-Kähler theorem guarantees that there exists a local foliation by integral manifolds.

Let θ^1, θ^2 be a basis of differential forms for $\mathcal{V}^\perp$. For a hyperbolic first order system the 2-dimensional integral elements of $\mathcal{V}$ are hyperbolic lines for the hyperbolic structure on $\mathcal{V}$. Locally we can choose two sections ϕ, ψ of $V_2(\mathcal{V}) \to M$ such that at each point $m \in M$ the two integral planes $\phi(m), \psi(m)$ are transversal. This means that $\mathcal{V}_m = \phi(m) \oplus \psi(m)$. We then choose 1-forms $\omega^1, \omega^2, \pi^1, \pi^2$ such that the forms $\theta^1, \theta^2, \omega^1, \omega^2, \pi^1, \pi^2$ are a basis for T^*M and ω^1, ω^2 vanish on ϕ and π^1, π^2 vanish on ψ. The structure equations for θ^1 and θ^2 are of the form

$$\mathrm{d}\theta^a = A^a_{\epsilon j}\pi^\epsilon \wedge \omega^j \mod \theta^1, \theta^2.$$

The terms $\omega^1 \wedge \omega^2$ and $\pi^1 \wedge \pi^2$ do not appear in these structure equations because the 2-planes $\phi(m)$ and $\psi(m)$ are integral elements for $\mathcal{V}$ and hence $\mathrm{d}\theta^1$ and $\mathrm{d}\theta^2$ vanish on the integral elements.

The integral manifolds of $\mathcal{V}$ are integral manifolds for the linear Pfaffian system defined by $I = \mathrm{span}(\theta^1, \theta^2)$, $J = \mathrm{span}(\theta^1, \theta^2, \omega^1, \omega^2)$. The structure equations show that the Pfaffian system has no intrinsic torsion. The basis transformations of θ^1, θ^2 and ω^1, ω^2 induce an action of the conformal group on the tableau defined by $A^a_{\epsilon i}$. This means that we can arrange in the hyperbolic case or elliptic case

$$\pi = \begin{pmatrix} \pi^1 & 0 \\ 0 & \pi^2 \end{pmatrix}, \quad \text{or} \quad \pi = \begin{pmatrix} \pi^1 & -\pi^2 \\ \pi^2 & \pi^1 \end{pmatrix}, \quad \text{respectively.}$$

Theorem 4.6.4 (Vessiot theorem for real-analytic first order systems). *Let M be a real analytic manifold of dimension 6 with a codimension 2 real-analytic distribution $\mathcal{V}$. If the Lie brackets modulo the subbundle $\lambda : \mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}$ are non-degenerate, then the system $(M, \mathcal{V})$ is locally contact equivalent to an elliptic or hyperbolic first order system of partial differential equations.*

Proof. The condition that the Lie brackets are non-degenerate implies that the system is either hyperbolic or elliptic. We assume the system is hyperbolic, the elliptic system can be treated in a similar way. The discussion above shows we can choose an adapted coframing

$\theta^1, \theta^2, \omega^1, \omega^2, \pi^1, \pi^2$ such that $I = \mathrm{span}(\theta^1, \theta^2) = \mathcal{V}^\perp$ and

$$\mathrm{d}\theta^1 \equiv -\pi^1 \wedge \omega^1 \mod I, \quad \mathrm{d}\theta^2 \equiv -\pi^2 \wedge \omega^2 \mod I.$$

This linear Pfaffian system has tableau of the form

$$\begin{pmatrix} \pi^1 & 0 \\ 0 & \pi^2 \end{pmatrix}.$$

This tableau is in involution and we can use a variation of the Cartan-Kähler theorem to conclude there exists a local foliation by integral manifolds of $\mathcal{V}$. The variation of the Cartan-Kähler theorem is described on pages 86–87 in Bryant et al. [13]. The foliation locally defines an integrable rank 2 subdistribution of $\mathcal{V}$ and a projection π to a base manifold. One can then continue as in the proof of the weak Vessiot theorem (4.6.3). □

The author's thesis advisor prof.dr. J.J. Duistermaat and the author have good reasons to believe the theorem is also true in the C^∞ setting. If we follow the proof of the theorem using the Cartan-Kähler theorem we arrive at a system of partial differential equations in Kowalevski form. The existence of solutions for this system in the C^∞ setting would prove the Vessiot theorem for first order systems. This system can be solved using the Kowalevski theorem, but this requires the structures to be analytic. The system of equations is a coupled system of an ordinary differential equation and a determined hyperbolic system. We believe that by setting up a contraction in an appropriate space we can prove existence of solutions. The author has checked that the theory of Yang [76] cannot be applied directly to the exterior differential system that is found when analyzing the conditions for existence of a local foliation by integral manifolds of the system.

Example 4.6.5 (First order wave equation). The first order system defined by

$$u_y = 0, \quad v_x = 0, \tag{4.15}$$

is called the *first order wave equation*. The reason for the name first order wave equation is that the system is the quotient of the wave equation $z_{xy} = 0$ under the symmetry ∂_z. Note that the first order wave equation is a *system* of partial differential equations and not a single equation.

If we use $x, y, u, v, p = u_x, s = v_y$ as coordinates for the equation manifold, then the Monge systems are given by

$$\mathcal{F} = \mathrm{span}\left(\partial_x + p\partial_u, \partial_p\right), \quad \mathcal{G} = \mathrm{span}\left(\partial_y + s\partial_v, \partial_s\right).$$

Both Monge systems have 3 invariants. ⊘

Example 4.6.6 (Cauchy-Riemann equations). Let B be a 4-dimensional manifold with an integrable almost complex structure $J^B : TB \to TB$. Since the almost complex structure is integrable we can introduce complex coordinates $x + iy$, $u + iv$ for the manifold B. The complex structure on the tangent space to B defined by J^B, is given in these coordinates by multiplication with i. A *holomorphic curve* in B is a real 2-dimensional surface for which the tangent space is complex-linear.

Every holomorphic curve can (after a change of coordinates) locally be written as the graph of the functions $u(x, y)$, $v(x, y)$. The condition that the graph of u and v defines a holomorphic curve in (B, J^B) is given by the *Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x. \tag{4.16}$$

Solutions of the Cauchy-Riemann equations are called holomorphic functions.

The Cauchy-Riemann equations form an elliptic first order system. The equations (4.16) define a codimension two submanifold M of $\mathrm{Gr}_2(TB)$. If we use $x, y, u, v, p = u_x, r = v_x$ as coordinates for M, then the contact distribution is given by

$$\mathcal{V} = \operatorname{span}\left(\partial_x + p\partial_u + r\partial_v, \partial_y - r\partial_u + p\partial_v, \partial_p, \partial_r\right).$$

The almost complex structure on M defined by the complex Monge systems is integrable. We can give M the complex coordinates $x + iy, u + iv, p + ir$. $\oslash$

4.6.2 The full Nijenhuis tensor

For second order equations we had the Cauchy characteristics $C(\mathcal{V}')$ that we could use to extend the Nijenhuis tensor. For first order systems $\mathcal{V}' = TM$ and hence the map J is defined on the entire tangent space so we can define the full Nijenhuis tensor $N : TM \to TM$.

We will start to make a classification of the first order systems depending on the structure of the image and kernel of the Nijenhuis tensor. Let $\mathcal{D} = \operatorname{im} N = \{ N(X, Y) \in TM \mid X, Y \in TM \}$ be the image of the Nijenhuis tensor. The Nijenhuis tensor is identically zero on $\mathcal{V} \times_M \mathcal{V}$ by definition. Also the bi-J-antilinear map on $(\mathcal{V}'/\mathcal{V}) \times_M (\mathcal{V}'/\mathcal{V})$ induced by J is equal to zero because the complex dimension of $\mathcal{V}'/\mathcal{V}$ is one and N is bi-J-antilinear. Together this implies the Nijenhuis tensor is already determined by the induced map

$$N : \mathcal{V} \times_M (\mathcal{V}'/\mathcal{V}) \to TM.$$

Recall that an element $Y \in \mathcal{V}'/\mathcal{V}$ is generic for J if $Y_+ \neq 0$ and $Y_- \neq 0$. For an elliptic system every non-zero $Y \in \mathcal{V}'/\mathcal{V}$ is generic. For a generic element $Y \in \mathcal{V}'/\mathcal{V}$ we define $A(Y) : TM \to TM : X \mapsto N(X, Y)$. The image $\mathcal{D}$ of N is equal to the image of A. The image of the Nijenhuis tensor can have rank 0, 1, 2, 3 or 4. In the elliptic case the image can only have rank 0, 2 or 4 (complex rank 0, 1 or 2).

Example 4.6.7. Consider the first order system depending on two parameters c_1, c_2 defined by the equations $u_y = c_1 v$ and $v_x = c_2 u$. We introduce coordinates $x, y, u, v, p = u_x, s = v_y$ for the equation manifold M. The Monge systems are given by

$$\begin{aligned}
\mathcal{F} &= \operatorname{span}\left(F_1 = \partial_x + p\partial_u + c_2 u\partial_v + c_1 c_2 v\partial_s, F_2 = \partial_p\right),\\
\mathcal{G} &= \operatorname{span}\left(G_1 = \partial_y + c_1 v\partial_u + s\partial_v + c_1 c_2 u\partial_p, G_2 = \partial_s\right).
\end{aligned}$$

The reader can check that the Monge systems satisfy the conditions $[\mathcal{F}, \mathcal{G}] \subset \mathcal{V}$ and $TM = \mathcal{F}' \oplus \mathcal{G}'$. Let $F_3 = [F_1, F_2] = -\partial_u$, $G_3 = [G_1, G_2] = -\partial_v$. With respect to the basis F_1, G_2, F_2, G_2, F_3, G_3 for TM the hyperbolic structure is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The image of N is spanned by $d_1 = N(F_1, F_3) = 4c_2\partial_v$ and $d_2 = N(G_1, G_3) = 4c_1\partial_u$. Depending on the values of the constants c_1 and c_2 the rank of $\mathcal{D}$ is 0, 1 or 2. $\oslash$

Generic image. The maximal rank of the image $\mathcal{D}$ of the Nijenhuis tensor is four. We will discuss the various possibilities for the Nijenhuis tensor in this case.

Lemma 4.6.8. *If* $\operatorname{rank} \mathcal{D} = 4$, *then the rank of* $\mathcal{D}/\mathcal{V}$ *in* $TM/\mathcal{V}$ *is equal to two.*

Proof. We give the proof for the elliptic case. The proof for the hyperbolic case can be done using Lemma 5.2.8 from Chapter 5. Since $\mathcal{D}/\mathcal{V} \subset TM/\mathcal{V}$ and $\operatorname{rank}(TM/\mathcal{V}) = 2$ the image $\mathcal{D}/\mathcal{V}$ can have rank at most two. If $\mathcal{D}/\mathcal{V} = 0$ then Lemma 4.6.14 shows that $\mathcal{D} = 0$. Hence if $\operatorname{rank} \mathcal{D} = 4$, then $\operatorname{rank}(\mathcal{D}/\mathcal{V}) \neq 0$. The image $\mathcal{D}$ is J-invariant and therefore the real rank of $\mathcal{D}/\mathcal{V}$ cannot be one and has to be two. □

Define $\mathcal{B}_1 = \{ X \in \mathcal{V} \mid A(Y)(X) \in \mathcal{V} \}$. It is easy to see that $\mathcal{B}_1$ does not depend on the element Y chosen (as long as Y is generic with respect to J). In terms of the Nijenhuis tensor $\mathcal{B}_1$ is equal to $\{ X \in \mathcal{V} \mid N(X, Y) \in V, \text{ for all } Y \in TM \}$. The rank of the distribution $\mathcal{B}_1$ is at least 2. We define $\mathcal{B}_2 = \mathcal{D} \cap \mathcal{V} = A(\mathcal{B}_1) \subset \mathcal{V}$.

The properties of the Nijenhuis tensor imply that both $\mathcal{B}_1$ and $\mathcal{B}_2$ are J-invariant. There are several possibilities: either $\mathcal{B}_1 = \mathcal{B}_2$, $\mathcal{V} = \mathcal{B}_1 \oplus \mathcal{B}_2$ or in the hyperbolic case we might have a mixed case. The case $\mathcal{V} = \mathcal{B}_1 \oplus \mathcal{B}_2$ is the generic situation and in Section 4.7.2 we construct an invariant framing on the equation manifold of the system.

We analyze the case $\mathcal{B}_1 = \mathcal{B}_2$.

Lemma 4.6.9. *Assume* $\operatorname{rank} \mathcal{D} = 4$. *If* $\mathcal{B}_1 = \mathcal{B}_2$, *then the bundle* $\mathcal{B}_1$ *is integrable.*

Proof. In the proof we need some theory from Chapter 5. The reader can skip the proof on first reading. The proof is for hyperbolic systems, for the elliptic systems there is a similar proof. The bundle $\mathcal{B}_1$ is an invariantly defined subbundle of $\mathcal{V}$ and hence we can choose an adapted coframing (5.14) with $\mathcal{B}_1$ as a distinguished subbundle. Using the structure group we can arrange that $T_{2^F} = 1$, $U_{2^F} = 0$, $V_{2^F} = 0$. Since $\mathcal{B}_1 = \mathcal{B}_2$ we have $U_{3^F} = 0$ and since $\operatorname{rank} \mathcal{D} = 4$ we have $V_{3^F} \in \mathbb{D}^*$. From Lemma 5.2.8 we find that $U_{3^F} + T_{2^F}(S_{3^F})^F = 0$ and hence $S_{3^F} = 0$. This proves that $\mathcal{B}_1$ is integrable. □

The integrable distribution $\mathcal{B}_1$ gives this class of systems a natural structure under point geometry. The author has the feeling that the class of equations for which $\mathcal{B}_1 = \mathcal{B}_2$ might be empty, although there is no proof yet. It is also unknown if there are any equations in the mixed case. The class of equations for which $\mathcal{V} = \mathcal{B}_1 \oplus \mathcal{B}_2$ contains the generic equations; an example is given below.

Example 4.6.10. Consider the hyperbolic first order system given by

$$u_y = (v_y)^2 + v, \quad v_x = (u_x)^2 + u.$$

We can define the following basis for the tangent space:

$$\begin{aligned}
F_1 &= \partial_x + p\partial_u + (p^2 + u)\partial_v + \partial_p - \left(\frac{s^2 + 2pu + 2p^3 + v}{H}\right)\partial_s,\\
F_2 &= \partial_p,\\
F_3 &= 2s\partial_u + \partial_v + 2\frac{-8ps^3 + 2pu + 2p^3 + v + 3s^2}{H^2}\partial_p\\
&\qquad + 6\frac{4vs^2 + 4s^2 + 2su - 2sp^2 + p}{H^2}\partial_s,\\
G_1 &= \partial_y + (s^2 + v)\partial_u + s\partial_v + \left(\frac{2sv + 2s^3 + u + p^2}{H}\right)\partial_p + \partial_s,\\
G_2 &= \partial_s,\\
G_3 &= 2\partial_u + 4p\partial_v + 12\frac{s - 2ps^2 + 4up^2 + 4p^4 + 2pv}{H^2}\partial_p\\
&\qquad + 4\frac{u + 3p^2 - 8p^3 s + 2s^3 + 2sv}{H^2}\partial_s,
\end{aligned}$$

with $H = 1 - 4ps$.

The distribution dual to the contact forms is $\mathcal{V} = \operatorname{span}(F_1, G_1, F_2, G_2)$, the almost product structure is defined by $\operatorname{span}(F_1, F_2, F_3) \oplus \operatorname{span}(G_1, G_2, G_3)$. The Nijenhuis tensor has generic image ($\operatorname{rank} \mathcal{D} = 4$, $\mathcal{B}_1 \cap \mathcal{B}_2 = 0$). We have $\mathcal{B}_1 = \operatorname{span}(F_1, G_1)$, $\mathcal{B}_2 = \operatorname{span}(F_2, G_2)$ and the image of $\mathcal{B}_2 \times_M TM$ under the Nijenhuis tensor is $\operatorname{span}(F_3, G_3)$. ⊘

4.6.3 Complexification of the tangent space

In this section we discuss the complexification $TM \otimes \mathbb{C}$ of the tangent space TM in more detail. In the elliptic setting this complexification can be used to define the complex characteristic systems of $\mathcal{V}$. With the complexification many proofs that work in the hyperbolic setting can be copied to the elliptic setting. We note that for this complexification we do not need the system to be analytic. In the 19th century a common technique was to complexify the entire manifold, for this construction we need M and $\mathcal{V}$ to be real analytic!

Let $(M, \mathcal{V})$ be an elliptic first order system. The almost complex structure on M will be denoted by J. We will start by characterizing the complex structure J in terms of subspaces of $TM \otimes \mathbb{C}$. Define

$$\mathcal{V}_\pm = \{\, X \in \mathcal{V} \otimes \mathbb{C} \mid JX = \pm i X \,\}. \tag{4.17}$$

The reader can check that the distributions $\mathcal{V}_\pm$ are equal to the previously defined Monge systems. The complexification $\mathcal{V} \otimes \mathbb{C}$ is equal to the direct sum $\mathcal{V}_+ \oplus \mathcal{V}_-$. This allows us to restrict the *projection operators*

$$\pi_\pm : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C} : X \mapsto (1/2)(1 \mp iJ)X. \tag{4.18}$$

to projection operators $\pi_\pm : \mathcal{V} \to \mathcal{V}_\pm$.

Let λ be the Lie brackets modulo $\mathcal{V}$. The complex-bilinearity of λ implies that $[\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V} \otimes \mathbb{C}$. Indeed, let $X \subset \mathcal{V}_+$, $Y \subset \mathcal{V}_-$. Then we have

$$\begin{aligned} 4[X, Y] &\equiv 4[\pi_+ X, \pi_- Y] \\ &\equiv [X - iJX, Y + iJY] \\ &\equiv [X, Y] + [JX, JY] - i[JX, Y] + i[X, JY] \\ &\equiv \lambda(X, Y) + J^2\lambda(X, Y) - iJ\lambda(X, Y) + iJ\lambda(X, Y) \\ &\equiv 0 \mod \mathcal{V}. \end{aligned}$$

The complex characteristic systems $\mathcal{V}_\pm$ have complex rank 2. By definition we have that $\mathcal{V}_\pm \subset (\mathcal{V}_\pm)'$. From this it follows that the bundles $(\mathcal{V}_\pm)'/\mathcal{V}_\pm$ have complex rank at most 1. The non-degeneracy of the Lie brackets implies that the complex rank of $(\mathcal{V}_\pm)'$ is indeed 3. Since $[\mathcal{V}, \mathcal{V}] = TM$ we have

$$\begin{aligned} TM \otimes \mathbb{C} = [\mathcal{V} \otimes \mathbb{C}, \mathcal{V} \otimes \mathbb{C}] &= [\mathcal{V}_+ + \mathcal{V}_-, \mathcal{V}_+ + \mathcal{V}_-] \\ &= [\mathcal{V}_+, \mathcal{V}_+] + [\mathcal{V}_-, \mathcal{V}_-] + [\mathcal{V}_+, \mathcal{V}_-] + [\mathcal{V}_-, \mathcal{V}_+] \\ &\subset (\mathcal{V}_+)' + (\mathcal{V}_-)' + (\mathcal{V} \otimes \mathbb{C}) \\ &\subset (\mathcal{V}_+)' + (\mathcal{V}_-)'. \end{aligned}$$

Since $\operatorname{rank}(\mathcal{V}_\pm)' = 3$ and $\operatorname{rank} TM \otimes \mathbb{C} = 6$ this implies $TM \otimes \mathbb{C} = (\mathcal{V}_+)' \oplus (\mathcal{V}_-)'$. We write $(\mathcal{V}')_\pm$ for the projections $\pi_\pm(\mathcal{V}' \otimes \mathbb{C}) = \pi_\pm(TM \otimes \mathbb{C})$. Since $(\mathcal{V}_\pm)' \subset TM_\pm$ (since λ is complex-bilinear) it follows from the ranks of the bundles that $(\mathcal{V}_\pm)' = (\mathcal{V}')_\pm$ and hence we can write unambiguously $\mathcal{V}'_\pm$ for $(\mathcal{V}_\pm)'$ or $(\mathcal{V}')_\pm$. For $X, Y \in TM$ we write $X = X_+ + X_-$ and $Y = Y_+ + Y_-$ with $X_\pm = \pi_\pm(X)$ and $Y_\pm = \pi_\pm(Y)$. On the pair X, Y the Nijenhuis tensor acts as

$$\begin{aligned} N(X, Y) &= [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] \\ &= -2([X_+, Y_+] + [X_-, Y_-]) - 2iJ([X_+, Y_+] - [X_-, Y_-]). \end{aligned}$$

We can rewrite this as

$$N(X, Y) = -4[X_+, Y_+]_- - 4[X_-, Y_-]_+. \tag{4.19}$$

For hyperbolic systems the hyperbolic structure K on the tangent space gives similar constructions, but we do not need to complexify. In the hyperbolic setting the characteristic systems are real and contained in the tangent bundle. In particular we can define $\mathcal{V}_+$ and $\mathcal{V}_-$ as the eigenspaces of $K : \mathcal{V} \to \mathcal{V}$ for the eigenvalues 1 and -1, respectively. We define projection operators

$$\pi_\pm : TM \to TM : X \mapsto (1/2)(1+K)X \tag{4.20}$$

and can write TM as the direct sum $\mathcal{V}'_+ \oplus \mathcal{V}'_-$. The Nijenhuis tensor acts on $X, Y \in TM$ as

$$N(X,Y) = 4[X_+, Y_+]_- + 4[X_-, Y_-]_+. \tag{4.21}$$

We will prove the following lemma both in the hyperbolic and the elliptic setting. In both settings a central role will be played by the characteristic systems. By using general properties of the characteristic systems the proof works for both first order systems and second order equations.

Lemma 4.6.11. *Let $(M, \mathcal{V})$ be a first order system (Definition 3.3.1) or second order scalar equation (a Vessiot system in two variables). Define $\mathcal{M}_\pm = \mathcal{V}'_\pm \oplus \mathcal{V}_\mp$. Then $[\mathcal{V}'_\pm, \mathcal{V}_\mp] = \mathcal{M}_\pm$.*

Proof. It is clear that both $\mathcal{V}'_\pm$ and $\mathcal{V}_\mp$ are contained in $[\mathcal{V}'_\pm, \mathcal{V}_\mp]$. So we need to prove that $[\mathcal{V}'_\pm, \mathcal{V}_\mp] \subset \mathcal{M}_\pm = \mathcal{V}'_\pm \oplus \mathcal{V}_\mp$.

Let $X, Y \subset \mathcal{V}_\pm$ and $Z \subset \mathcal{V}_\mp$. From the Jacobi identity we have

$$[[X,Y],Z] = [X,[Y,Z]] + [[X,Z],Y].$$

We use that $[\mathcal{V}_\pm, \mathcal{V}_\mp] \subset \mathcal{V}$ in the hyperbolic setting and $[\mathcal{V}_\pm, \mathcal{V}_\mp] \subset \mathcal{V} \otimes \mathbb{C}$ in the elliptic setting. We write $\mathcal{V}^{\mathrm{HE}}$ for either $\mathcal{V}$ or $\mathcal{V} \otimes \mathbb{C}$ depending on the context. Then

$$[[X,Y],Z] \subset [\mathcal{V}_\pm, \mathcal{V}^{\mathrm{HE}}] + [\mathcal{V}^{\mathrm{HE}}, \mathcal{V}_\pm] \subset \mathcal{M}_\pm.$$

Since the vector fields $[X,Y]$ with $X, Y \subset \mathcal{V}_\pm$ span $\mathcal{V}'_\pm$ this concludes the proof. ☐

4.6.4 Čap and Eastwood

In the paper *Some special geometry in dimension six* [18] Čap and Eastwood study the geometry of a codimension two distribution on a manifold of dimension 6. In the elliptic case they use the distribution to construct a unique complex structure, for the hyperbolic case they have similar results. The remarkable thing is that their construction of the almost complex structure is different from our construction. The structure of Čap and Eastwood and our structure turns out to be the same, but the proof is non-trivial.

Let (M, H) be a smooth connected manifold M of dimension 6 together with a codimension two distribution H and an orientation on M. Čap and Eastwood find (just like we did in Section 4.2) that the Lie brackets define an anti-symmetric bilinear map $H \times_M H \to TM/H$, or a vector bundle map $\mathcal{L} : H \wedge H \to TM/H$. They define the system to be elliptic or hyperbolic in the case that $\mathcal{L} \wedge \mathcal{L}$ as a tensor in $\Gamma(\Lambda^4 H^* \otimes S^2(TM/H))$ is non-degenerate definite or non-degenerate indefinite, respectively. This characterization corresponds to our formulation in Section 4.2. Then they prove there exists a unique almost complex structure on M.

Theorem 4.6.12 (Theorem on page 2 in Čap and Eastwood [18]). *Suppose* (M, H) *is elliptic. Then* M *admits a unique almost complex structure* $J_{\mathrm{CE}} : TM \to TM$ *characterized by the following properties:–*

i) J_{CE} *preserves* H*;*

ii) the orientation on M *induced by* J_{CE} *is the given one;*

iii) $\mathcal{L} : H \times_M H \to TM/H$ *is complex-bilinear for the induced structures, or equivalently* $[\xi, \eta] + J_{\mathrm{CE}}[J_{\mathrm{CE}}\xi, \eta] \in \Gamma(H)$ *for* $\xi, \eta \in \Gamma(H)$*;*

iv) $[\xi, \eta] + J_{\mathrm{CE}}[J_{\mathrm{CE}}\xi, \eta] - J_{\mathrm{CE}}[\xi, J_{\mathrm{CE}}\eta] + [J_{\mathrm{CE}}\xi, J_{\mathrm{CE}}\eta] \in \Gamma(H)$ *for* $\xi \in \Gamma(TM)$, $\eta \in \Gamma(H)$.

Furthermore, the tensor $S : TM/H \otimes H \to TM/H$ *induced by*

$$S(\xi, \eta) = [\xi, \eta] + J_{\mathrm{CE}}[J_{\mathrm{CE}}\xi, \eta] \mod H \text{ for } \xi \in \Gamma(TM),\ \eta \in \Gamma(H)$$

is the obstruction to J_{CE} *being integrable.*

Let us analyze this theorem step by step by comparing it to our construction of the almost complex structure. The first condition that J_{CE} preserves the distribution H is in the definition of our almost complex structure. We have not chosen an orientation on M, so we do not have the second property in the theorem. However, choosing a different orientation would only give J_{CE} a different sign. The complex-bilinearity of J_{CE} follows from our definition of J on TM/H, see Section 4.2. The equivalent formulation of Čap and Eastwood is in terms of sections of H, but one can easily check that the conditions depend only on the value of the section at a particular point and not on the derivatives.

Finally we have to analyze the last condition in the theorem. The expression

$$F = [\xi, \eta] + J_{\mathrm{CE}}[J_{\mathrm{CE}}\xi, \eta] - J_{\mathrm{CE}}[\xi, J_{\mathrm{CE}}\eta] + [J_{\mathrm{CE}}\xi, J_{\mathrm{CE}}\eta] \tag{4.22}$$

looks like the Nijenhuis tensor, but is not a true tensor. The expression F depends on the first order derivatives of η. The Nijenhuis tensor on the other hand is a true tensor, i.e., the expression

$$N(\xi, \eta) = J^2[\xi, \eta] - J[J\xi, \eta] - J[\xi, J\eta] + [J\xi, J\eta] \tag{4.23}$$

at a point $m \in M$ only depends on the values ξ_m and η_m.

We will use the complexification of the tangent bundle to prove that the almost complex structure defined by Čap and Eastwood is identical to our almost complex structure.

Theorem 4.6.13. *The almost complex structure* J_{CE} *defined in Theorem 4.6.12 is equal to the almost complex structure* J *defined in Theorem 4.6.1.*

Proof. In the analysis above we have already shown that the almost complex structure J satisfies the conditions (i)–(iii) in Theorem 4.6.12. It remains to show that J satisfies the final condition (iv). The uniqueness of the almost complex structure J_{CE} then implies $J = J_{\mathrm{CE}}$.

Let X, Y be vector fields in TM with $Y \subset \mathcal{V}$. Write $X = X_+ + X_-$, $Y = Y_+ + Y_-$ with $X_\pm \subset \mathcal{V}'_\pm$, $Y_\pm \subset \mathcal{V}_\pm$. The expression in the fourth condition of Čap and Eastwood is

$$\begin{aligned}
&[X, Y] + J[JX, Y] - J[X, JY] + [JX, JY] \\
&\quad = [X_+ + X_-, Y_+ + Y_-] + iJ[X_+ - X_-, Y_+ + Y_-] \\
&\qquad - iJ[X_+ + X_-, Y_+ + Y_-] - [X_+ - X_-, Y_+ - Y_-] \\
&\quad = [X_+ + X_-, Y_+ + Y_-] + iJ[X_+ - X_-, Y_+ + Y_-] \\
&\qquad - iJ[X_+ + X_-, Y_+ - Y_-] - [X_+ - X_-, Y_+ - Y_-] \\
&\quad = 2([X_+, Y_-] + [X_-, Y_+]) + 2iJ([X_+, Y_-] - [X_-, Y_+]) \\
&\quad = 2[X_+, Y_-]_- + 2[X_-, Y_+]_+.
\end{aligned}$$

From Lemma 4.6.11 we know that $[X_+, Y_-] \subset \mathcal{V}'_+ \oplus \mathcal{V}_-$ and hence $[X_+, Y_-]_- \subset \mathcal{V}_- \subset \mathcal{V} \otimes \mathbb{C}$. In the same way $[X_-, Y_+]_+ \subset \mathcal{V} \otimes \mathbb{C}$. Together this yields

$$[X, Y] + J[JX, Y] - J[X, JY] + [JX, JY] = 2[X_+, Y_-]_- + 2[X_-, Y_+]_+ \subset \mathcal{V} \otimes \mathbb{C}.$$

We conclude that our J satisfies the defining conditions of the almost complex structure of Čap and Eastwood. □

In the article Čap and Eastwood also define an almost product structure for the hyperbolic systems. Their almost product structure in the hyperbolic case is identical to our almost product structure for hyperbolic first order systems.

4.6.5 The flat case

If the Nijenhuis tensor is identically zero, then there is a unique complex structure on the manifold M and the manifold is complex-analytic. This is the Newlander-Nirenberg theorem, first proved in Newlander and Nirenberg [57].

Lemma 4.6.14. *Let $(M, \mathcal{V})$ be a hyperbolic or elliptic first order system with Nijenhuis tensor N. If $N(TM \times_M TM) \subset \mathcal{V}$ then $N = 0$.*

Proof. We give the proof for elliptic systems. The proof for hyperbolic systems is identical with the simplification that we do not need to complexify. The rank of a bundle is the complex rank for elliptic systems and the real rank for hyperbolic systems.

Let $X = X_+ + X_-$, $Y = Y_+ + Y_-$ be sections of $TM \otimes \mathbb{C}$, with X_+, X_- and Y_+,Y_- the components of X and Y, respectively, in TM_+ and TM_-. From equation (4.19) we see that $\operatorname{im} N \subset \mathcal{V}$ is equivalent to $[\mathcal{V}_\pm, \mathcal{V}'_\pm] \subset \mathcal{M}_\pm$. We always have $[\mathcal{V}_\pm, \mathcal{V}_\mp] \subset \mathcal{V} \otimes \mathbb{C} \subset \mathcal{M}_\pm$. Consider the Cauchy characteristic system $C(\mathcal{M}_\pm)$. From the above we have $\mathcal{V}_\pm \subset C(\mathcal{M}_\pm)$. Since Cauchy characteristic systems are integrable we also have $\mathcal{V}'_\pm \subset C(\mathcal{M}_\pm)$.

The space $C(\mathcal{M}_\pm)$ is equal to the kernel of the Lie brackets modulo $\mathcal{M}_\pm$. Because the corank of $\mathcal{M}_\pm$ in $\mathcal{V}' = TM$ is 1 the Lie brackets modulo $\mathcal{M}_\pm$ define a conformal bilinear anti-symmetric form on $\mathcal{M}_\pm$. The kernel of an anti-symmetric bilinear form has even codimension and hence $\operatorname{rank} C(\mathcal{M}_\pm) = 1, 3, 5$. We already know that $[\mathcal{M}_\pm, \mathcal{M}_\pm] = \mathcal{V}'$ so

$\mathcal{V}'_\pm \subset C(\mathcal{M}_\pm) \neq \mathcal{M}_\pm$. Since rank $\mathcal{V}'_\pm = 3$ we conclude that $C(\mathcal{M}_\pm)$ must have rank 3 and hence $C(\mathcal{M}_\pm) = \mathcal{V}'_\pm$. Therefore $\mathcal{V}'_\pm$ is integrable, so $[\mathcal{V}'_\pm, \mathcal{V}'_\pm] \subset \mathcal{V}'_\pm$. From equation (4.19) we conclude that $N = 0$. □

Remark 4.6.15. The statement of the lemma is equivalent to the last part of Theorem 4.6.12. The vanishing of the tensor S introduced in that theorem is equivalent to the condition that N takes values in $\mathcal{V}$. In terms of differential forms the lemma can also be proved, see Lemma 5.2.8. ⊘

Proposition 4.6.16. *Let M be a complex-analytic manifold and $\mathcal{V}$ a complex-linear distribution on M. By complex-linear we mean that $J\mathcal{V} \subset \mathcal{V}$. We assume that the Lie brackets modulo $\mathcal{V}$ define a complex-bilinear map from $\mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}$ and that $[\mathcal{V}, \mathcal{V}]$ spans TM. Then $\mathcal{V}$ is holomorphic.*

Proof. Let $\mathcal{V}$ have complex dimension n and complex codimension c in M. Since M is complex-analytic we can assume there are complex coordinates $x^1, \ldots, x^n, z^1, \ldots, z^c$ for M such that the ideal I dual to the distribution $\mathcal{V}$ is spanned by

$$\theta^i = \mathrm{d}z^i - \sum_{\mu=1}^{n} a^i_\mu \mathrm{d}x^\mu, \; i = 1, \ldots, n.$$

We will use the summation convention where roman indices will run from 1 to n and Greek indices will run from 1 to c. The exterior derivative of θ^i is given by

$$\begin{aligned}
\mathrm{d}\theta^i &\equiv -\mathrm{d}a^i_\alpha \wedge \mathrm{d}x^\alpha \\
&\equiv -\left(\frac{\partial a^i_\alpha}{\partial x^\beta} \mathrm{d}x^\beta \wedge \mathrm{d}x^\alpha + \frac{\partial a^i_\alpha}{\partial \bar{x}^\beta} \mathrm{d}\bar{x}^\beta \wedge \mathrm{d}x^\alpha + \frac{\partial a^i_\alpha}{\partial z^j} \mathrm{d}z^j \wedge \mathrm{d}x^\alpha + \frac{\partial a^i_\alpha}{\partial \bar{z}^j} \mathrm{d}\bar{z}^j \wedge \mathrm{d}x^\alpha \right) \\
&\equiv -\left(\frac{\partial a^i_\alpha}{\partial x^\beta} + a^j_\beta \frac{\partial a^i_\alpha}{\partial z^j} \right) \mathrm{d}x^\beta \wedge \mathrm{d}x^\alpha - \left(\frac{\partial a^i_\alpha}{\partial \bar{x}^\beta} + \bar{a}^j_\beta \frac{\partial a^i_\alpha}{\partial \bar{z}^j} \right) \mathrm{d}\bar{x}^\beta \wedge \mathrm{d}x^\alpha \\
&\equiv S^i_{\alpha,\beta} \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta + T^i_{\alpha,\beta} \mathrm{d}x^\alpha \wedge \mathrm{d}\bar{x}^\beta \quad \mod I,
\end{aligned}$$

with $S^i_{\alpha,\beta} = -S^i_{\beta,\alpha}$,

$$\begin{aligned}
S^i_{\alpha,\beta} &= \frac{1}{2} \left(\frac{\partial a^i_\alpha}{\partial x^\beta} + a^j_\beta \frac{\partial a^i_\alpha}{\partial z^j} - \frac{\partial a^i_\beta}{\partial x^\alpha} - a^j_\alpha \frac{\partial a^i_\beta}{\partial z^j} \right), \\
T^i_{\alpha,\beta} &= \frac{\partial a^i_\alpha}{\partial \bar{x}^\beta} + \bar{a}^j_\beta \frac{\partial a^i_\alpha}{\partial \bar{z}^j}.
\end{aligned}$$

Since the Lie brackets modulo $\mathcal{V}$ are complex-bilinear and non-degenerate we must have

$T^i_{\alpha,\beta} = 0$ for all i, α, β and for every i there must be a non-zero $S^i_{\alpha,\beta}$. We calculate

$$
\begin{aligned}
0 &= \frac{\partial}{\partial \bar{x}^\alpha} T^i_{\mu,\beta} = \frac{\partial}{\partial \bar{x}^\alpha}\left(\frac{\partial a^i_\mu}{\partial \bar{x}^\beta} + \bar{a}^j_\beta \frac{\partial a^i_\mu}{\partial \bar{z}^j}\right)\\
&= \frac{\partial}{\partial \bar{x}^\beta}\left(\frac{\partial a^i_\mu}{\partial \bar{x}^\alpha}\right) + \frac{\partial \bar{a}^j_\beta}{\partial \bar{x}^\alpha}\frac{\partial a^i_\mu}{\partial \bar{z}^j} + \bar{a}^j_\beta \frac{\partial^2 a^i_\mu}{\partial \bar{x}^\alpha \partial \bar{z}^j}\\
&= \frac{\partial}{\partial \bar{x}^\beta}\left(-\bar{a}^j_\alpha \frac{\partial a^i_\mu}{\partial \bar{z}^j}\right) + \frac{\partial \bar{a}^j_\beta}{\partial \bar{x}^\alpha}\frac{\partial a^i_\mu}{\partial \bar{z}^j} + \bar{a}^j_\beta \frac{\partial}{\partial \bar{z}^j}\left(-\bar{a}^k_\alpha \frac{\partial a^i_\mu}{\partial \bar{z}^k}\right)\\
&= -\frac{\partial \bar{a}^j_\alpha}{\partial \bar{x}^\beta}\frac{\partial a^i_\mu}{\partial \bar{z}^j} - \bar{a}^j_\alpha \frac{\partial^2 a^i_\mu}{\partial \bar{x}^\beta \partial \bar{z}^j} + \overline{\frac{\partial a^j_\beta}{\partial x^\alpha}}\frac{\partial a^i_\mu}{\partial \bar{z}^j} - \bar{a}^j_\beta \frac{\partial \bar{a}^k_\alpha}{\partial \bar{z}^j}\frac{\partial a^i_\mu}{\partial \bar{z}^k} - \bar{a}^j_\beta \bar{a}^k_\alpha \frac{\partial^2 a^i_\mu}{\partial \bar{z}^j \partial \bar{z}^k}\\
&= \overline{\left(-\frac{\partial a^j_\alpha}{\partial x^\beta} + \frac{\partial a^j_\beta}{\partial x^\alpha} - a^k_\beta \frac{\partial a^j_\alpha}{\partial z^k}\right)}\frac{\partial a^i_\mu}{\partial \bar{z}^j} - \bar{a}^j_\alpha \frac{\partial}{\partial \bar{z}^j}\left(\frac{\partial a^i_\mu}{\partial \bar{x}^\beta}\right) - \bar{a}^k_\beta \bar{a}^j_\alpha \frac{\partial^2 a^i_\mu}{\partial \bar{z}^k \partial \bar{z}^j}\\
&= \overline{\left(-\frac{\partial a^j_\alpha}{\partial x^\beta} + \frac{\partial a^j_\beta}{\partial x^\alpha} - a^k_\beta \frac{\partial a^j_\alpha}{\partial z^k}\right)}\frac{\partial a^i_\mu}{\partial \bar{z}^j} + \bar{a}^j_\alpha \frac{\partial}{\partial \bar{z}^j}\left(\bar{a}^k_\beta \frac{\partial a^i_\mu}{\partial \bar{z}^k}\right) - \bar{a}^j_\alpha \bar{a}^k_\beta \frac{\partial^2 a^i_\mu}{\partial \bar{z}^k \partial \bar{z}^j}\\
&= \overline{\left(-\frac{\partial a^j_\alpha}{\partial x^\beta} + \frac{\partial a^j_\beta}{\partial x^\alpha} - a^k_\beta \frac{\partial a^j_\alpha}{\partial z^k} + a^k_\alpha \frac{\partial a^j_\beta}{\partial z^k}\right)}\frac{\partial a^i_\mu}{\partial \bar{z}^j}\\
&= -2 S^i_{\alpha,\beta}\frac{\partial a^i_\mu}{\partial \bar{z}^j}.
\end{aligned}
$$

The condition that $[\mathcal{V}, \mathcal{V}]$ spans TM implies that for every index i we can select a pair α, β such that $S^i_{\alpha,\beta} \neq 0$. It follows that $\partial a^i_\mu / \partial \bar{z}^j = 0$ for all i, j, μ. From the fact that $T^i_{\alpha,\beta} = 0$ it then follows that $\partial a^i_\mu / \partial \bar{x}^\alpha$ for all i, α, μ as well. The functions a^i_μ are holomorphic and hence $\mathcal{V}$ is holomorphic. □

Remark 4.6.17. The cases where $\mathcal{V}$ has complex dimension 0 or 1 or complex codimension 0 are all trivial. The lowest dimensional non-trivial example is therefore the case where $\dim_{\mathbb{C}} M = 3$ and $\dim_{\mathbb{C}} \mathcal{V} = 2$. This case is precisely the case of the Cauchy-Riemann equations (see Example 4.6.6). ⊘

Example 4.6.18. The condition that $[\mathcal{V}, \mathcal{V}]$ spans TM is necessary. Consider the manifold M of complex dimension 4 with complex coordinates $x = x^1 + ix^2$, $y = y^1 + iy^2$, $z = z^1 + iz^2$, $p = p^1 + ip^2$ and the complex 1-forms $\alpha = \mathrm{d}z - (y + \bar{p})\mathrm{d}x$, $\beta = \mathrm{d}p$. The distribution dual to these 1-forms is given in real coordinates by

$$
\begin{gathered}
\partial_{x^1} + (y^1 + p^1)\partial_{z^1} + (y^2 - p^2)\partial_{z^2},\ \partial_{y^1},\\
\partial_{x^2} + (-y^2 + p^2)\partial_{z^1} + (y^1 + p^1)\partial_{z^2},\ \partial_{y^2}.
\end{gathered}
$$

The distribution $\mathcal{V}$ satisfies all the conditions for the theorem, except that $[\mathcal{V}, \mathcal{V}] \neq TM$. The distribution $\mathcal{V}$ is not holomorphic. ⊘

There is also a version of Proposition 4.6.16 for almost product structures.

Proposition 4.6.19. *Let M be a direct product manifold $M_1 \times M_2$ with hyperbolic structure K and $\mathcal{V}$ a distribution on M that is K-invariant. We assume that the Lie brackets modulo $\mathcal{V}$ define a K-bilinear map from $\mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}$ and that $[\mathcal{V}, \mathcal{V}]$ spans TM. Then $\mathcal{V}$ is splits into two distributions $\mathcal{V}_1$ and $\mathcal{V}_2$ on M_1 and M_2, respectively, and $\mathcal{V}$ is equal to the direct sum of $\mathcal{V}_1$ and $\mathcal{V}_2$.*

First order systems for which the almost complex structure is integrable are complex analytic systems. Similarly, if the system is hyperbolic with integrable almost product structure, then the system M is a direct product $M_1 \times M_2$. On each component M_j we have a contact structure. Since all contact structures are equivalent under contact transformations all flat hyperbolic systems are equivalent to the direct product of two contact structures. An example in local coordinates is the first order wave equation, see Example 4.6.5.

4.6.6 Summary

Let $(M, \mathcal{V})$ be an elliptic first order system. Write $J_{\mathcal{V}}$ and $J_{TM/\mathcal{V}}$ for the complex structures on $\mathcal{V}$ and $\mathcal{V}'/\mathcal{V}$ that make the Lie brackets modulo the subbundle complex-bilinear (see Section 4.2).

Let J be an almost complex structure on M such that $J\mathcal{V} \subset \mathcal{V}$, $J|_{\mathcal{V}} = J_{\mathcal{V}}$, and the by J induced endomorphism on $TM/\mathcal{V}$ equal to $J_{TM/\mathcal{V}}$. Let $T_\pm$ be the eigenspaces in $TM \otimes \mathbb{C}$ of the operator J for the eigenvalues $\pm$. Define the tensor

$$C : \mathcal{V} \times_M (TM/\mathcal{V}) \to TM/\mathcal{V} : (X, Y) \mapsto [X, JY] - J[X, Y] \mod \mathcal{V}.$$

Then the following statements are equivalent.

a) The Nijenhuis tensor of J is identically zero on $\mathcal{V} \times_M \mathcal{V}$.

b) The almost complex structure J equals the almost complex structure of Čap and Eastwood, in other words C is J-linear with respect to the first variable.

c) $T_\pm = \mathcal{V}'_\pm$.

d) $[\mathcal{V}_\pm, T_\mp] \subset \operatorname{span}(T_\mp, \mathcal{V}_\pm)$.

Let J satisfy one of the equivalent conditions above. Then the following four statements are equivalent.

i) $N(\mathcal{V} \times_M TM) \subset \mathcal{V}$.

ii) $C = 0$.

iii) $[\mathcal{V}_\pm, \mathcal{V}'_\pm] \subset \operatorname{span}(\mathcal{V}'_\pm, \mathcal{V}_\mp)$.

iv) $N = 0$.

4.7 Invariant framings and orders for first order systems

In this section we study elliptic and hyperbolic first order systems under contact geometry. Such a system is given by a 6-dimensional manifold with a rank 4 distribution on which the Lie brackets modulo the subbundle are non-degenerate. We will give a summary of the various structures on the equation manifold we have introduced in the previous sections and keep track of the order of the various structures. Then we will give for the generic systems a framing that is invariant under the transformations leaving invariant the contact distribution. With the invariant framing we will be able to define two invariants for first order systems.

4.7.1 Orders

The structure of the manifold, i.e., the distribution $\mathcal{V}$ is by definition of order zero. The structure introduced by the Lie brackets modulo the subbundle (such as the Monge systems and the complex or hyperbolic structure J on $\mathcal{V}$) is of first order since we have to differentiate the distribution once to calculate the Lie brackets modulo the subbundle.

For a given bundle $\mathcal{W}$ we can define the Lie brackets modulo the subbundle $\lambda = [\cdot, \cdot]_{/\mathcal{W}} : \mathcal{W} \times_M \mathcal{W} \to TM/\mathcal{W}$. This is a tensor and hence defines at every point a structure on the manifold M. The order of this tensor is equal to the order of the bundle $\mathcal{W}$ plus one. The fact that the degree of the tensor is one order higher than the order of $\mathcal{W}$ follows from the fact that we need to know the derivative of $\mathcal{W}$ at a point x in order to calculate $\lambda_x : \mathcal{W}_x \times \mathcal{W}_x \to T_x M/\mathcal{W}_x$.

Example 4.7.1. On $\mathbb{R}^3$ with coordinates x, y, z define the distributions $\mathcal{V}$ and $\mathcal{W}$ by

$$\mathcal{V} = \operatorname{span}\left(\partial_x, \partial_y\right), \quad \mathcal{W} = \operatorname{span}\left(\partial_x, \partial_y + x\partial_z\right).$$

At the origin the bundles $\mathcal{V}$ and $\mathcal{W}$ are identical and $V = \mathcal{V}_0 = \mathcal{W}_0$ is spanned by the vectors $e_1 = (\partial_x)_0$ and $e_2 = (\partial_y)_0$. The Lie brackets modulo the subbundle are given at the origin by

$$[\cdot, \cdot]_{/\mathcal{V}} : V \times V \to \mathbb{R}^3/V : (e_1, e_2) \mapsto 0 \mod V,$$
$$[\cdot, \cdot]_{/\mathcal{W}} : V \times V \to \mathbb{R}^3/V : (e_1, e_2) \mapsto (\partial_z)_0 \mod V. \qquad \oslash$$

Remark 4.7.2. Our definition of order might be a bit confusing. The distribution $\mathcal{V}$ is in our definition a structure of order zero, but the distribution is defined in the tangent space of the equation manifold and hence is expressed in the first order derivatives of the equation manifold M. In other words, the distribution is an expression in the coordinates for the $\mathrm{Gr}_2(TM)$ and these coordinates are first order in the coordinates of the underlying manifold M.

When we have a base manifold B for M things may be even more confusing. The manifold M is a submanifold of $\mathrm{Gr}_2(TB)$ where B is a base manifold for M. Hence the distribution is a second order expression in the coordinates for B. $\oslash$

For each order 0, 1, 2, 3, 4 we give a description of the structures we have found so far.

Order 0. We are given the manifold M and $\mathcal{V} \subset TM$. These two structures contain all the information at this order. Unless we define some additional structure (such as a base manifold for M) there is no additional structure to be found.

Order 1. By taking Lie brackets modulo $\mathcal{V}$ we find a unique complex structure or hyperbolic structure on $\mathcal{V}$ and on $TM/\mathcal{V}$ (up to a choice of orientation, or sign of the complex structure). With these structures we have a complex-linear or hyperbolic-linear quadratic form $\mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}$. At this order this is all the structure we can find.

Order 2. We can extend the structures found above to an almost complex structure or almost product structure on the entire tangent space. Recall that we can make a unique choice of almost complex structure or almost product structure by requiring the Nijenhuis tensor to vanish on $\mathcal{V} \times_M \mathcal{V}$. Note that the Nijenhuis tensor is a third order object, but the restriction of the Nijenhuis tensor to $\mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}$ is only a second order object!

At this order we also have the tensors $C : \mathcal{V} \times_M TM/\mathcal{V} \to TM/\mathcal{V} : (X, Y) \mapsto [X, JY] - J[X, Y]$ and $N : \mathcal{V} \times_M TM/\mathcal{V} \to TM/\mathcal{V} : (X, Y) \mapsto C(JX, Y) - JC(X, Y) \mod \mathcal{V}$. The bundle $\mathcal{B}_1$ is also of order 2 since $\mathcal{B}_1$ is equal to the intersection of $\mathcal{V}$ with the kernel of the map $N : \mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}$.

Order 3. Using the full almost complex structure we can calculate the full Nijenhuis tensor. The distribution $\mathcal{B}_2$ is of order 3 since $\mathcal{B}_2 = N(\mathcal{B}_1, TM)$. The image $\mathcal{B}_3 = N(\mathcal{B}_2, TM)$ of $\mathcal{B}_2$ is of order 3 as well.

Order 4. In the generic setting we expect many invariants at order 4. At order 3 we can construct an invariant coframing and the structure functions of the invariant coframe are all continuous invariants (up to a discrete symmetry). Many of the structure functions will be trivial, but not all of them.

4.7.2 The invariant framing

Depending on the system the distributions $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{D}$ can have different ranks. To make full use of the structure present at a certain order we have to make some assumptions on the ranks of the different distributions.

The rank of the image $\mathcal{D}$ of N is an obvious choice to consider. For an elliptic system the Nijenhuis tensor is anti-symmetric conjugate-bilinear and hence the image $\mathcal{D}$ has real rank 0, 2 or 4. If the image has rank 0 the system is a complex contact manifold of complex dimension 3, see Section 4.6.5. For rank 4 we can construct an invariant coframing under some additional assumptions, see the paragraphs below. For rank 2 we have too little freedom to construct a normal form for the system but too much freedom to create an invariant framing. In Section 7.1 we will return to the systems with rank $\mathcal{D} = 2$.

For generic elliptic systems (Nijenhuis tensor has image of dimension 4 and $\mathcal{B}_1 \neq \mathcal{B}_2$) we can proceed as follows. The distribution $\mathcal{B}_1$ is a complex 1-dimensional and J-invariant subspace of $\mathcal{V}$. The image $\mathcal{B}_2 = N(\mathcal{B}_1, TM)$ is a complex 1-dimensional invariant subspace of $\mathcal{V}$. In the generic situation $\mathcal{B}_1 \neq \mathcal{B}_2$. We will assume from here on this is the case. Then $\mathcal{B}_3 = N(\mathcal{B}_2, TM)$ is a rank 2 distribution and $TM = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \mathcal{B}_3$. Choose a vector field $e_3 \subset \mathcal{B}_3$ and define $A : TM \to TM : X \mapsto N(X, e_3)$. Pick $e_1 \subset \mathcal{B}_1$ and $e_2 \subset \mathcal{B}_2$. The triple (e_1, e_2, e_3) is unique up to multiplications with complex scalar factors. We have

$$\begin{aligned} [e_1, e_2] &\equiv \alpha e_3 \mod \mathcal{V}, \\ A(e1) &= \beta e_2, \\ A(e_2) &= \gamma e_3, \end{aligned}$$

for certain complex functions α, β, γ.

We can scale the representative vectors e_j by complex numbers ϕ, ψ, χ to $\tilde{e}_1 = \phi e_1$, $\tilde{e}_2 = \psi e_2$, $\tilde{e}_3 = \chi e_3$. We find new coefficients

$$\tilde{\alpha} = \phi\psi\alpha/\chi, \quad \tilde{\beta} = \bar{\phi}\bar{\chi}\beta/\psi, \quad \tilde{\gamma} = \bar{\psi}\bar{\chi}\gamma/\chi.$$

We can normalize these all to 1, we find that

$$\begin{aligned} \phi^2 &= \beta\bar{\gamma}/(\bar{\alpha}\bar{\beta}^2\gamma), \\ \psi &= \phi\bar{\phi}\bar{\alpha}\bar{\beta}/\bar{\gamma}, \\ \chi &= \phi\psi\alpha. \end{aligned}$$

After the normalization we are left with a discrete symmetry to transform $e_1 \to \pm e_1, e_2 \mapsto e_2, e_3 \mapsto e_3$. Hence every generic elliptic first order system has a up to a discrete symmetry a unique invariant framing. The invariant framing is of order 3.

4.7.3 The hyperbolic case

In the hyperbolic case similar statements hold, but we need some more assumptions on the generality of the system. For example in the hyperbolic setting the image of the Nijenhuis tensor can be 0, 1, 2, 3, or 4 dimensional. We can apply the same theory, but work with hyperbolic variables $(x, y)^T$ instead of complex variables $x + iy$. When we choose a representative element in $\mathcal{B}_1, \mathcal{B}_2$ or $A(\mathcal{B}_2)$ we must always make a generic choice with respect to the almost product structure, i.e., take an element X such that X and JX are linearly independent and hence span a 2-dimensional space. If X is picked in $TM_\pm$, then $JX = \pm X$ and hence X and JX are not linearly independent. An element X is generic for the almost product structure if and only of $X_+ \neq 0$ and $X_- \neq 0$. Another problem can arise when we have to normalize the coframe. The hyperbolic formula for ϕ is $\phi^2 = \beta\gamma^F/(\alpha^F(\beta^F)^2\gamma)$. The right hand side is not necessarily positive, and the square root could be imaginary. We can solve this by either working with a complex coframe or normalizing β and γ to 1 and α to $(1, 1)^T$, $(1, -1)^T$, $(-1, 1)^T$ or $(-1, -1)^T$.

4.8 Continuous invariants

The structure theory developed so far has not provided us with any continuous invariants. For example in Riemannian geometry the scalar curvature is a continuous invariant (of second order in the metric). For our systems we have found invariant structures, such as the almost complex structure, the image and kernel of the Nijenhuis tensor. In this section we will show that at first and second order there are no continuous local invariants. At the third order there are precisely two continuous invariants.

To analyze the continuous local invariants we make use of the theory of jet groupoids and natural bundles. This theory [8] was explained to the author by Mohamed Barakat. The existence of invariants at order three is confirmed by a geometric construction of the invariants in Section 4.8.2.

4.8.1 Jet groupoids

The diffeomorphism group of a smooth manifold acts on every geometric structure on the manifold. If we are interested in the action of the diffeomorphism group on the finite order part of a structure, then we only need to study the finite order jets of diffeomorphisms. A problem with these finite order jets is that, while the jets of diffeomorphisms fixing a point form a group, the jets of general diffeomorphisms do not form a group. Let $j^r_x \phi_0$ and $j^r_y \phi_1$ be two jets of diffeomorphisms based at the points x and y. We can only make the composition $j^r_y \phi_1 \circ j^r_x \phi_0$ if $\phi_0(x) = y$. For this reason the jets of smooth functions do not form a group. The jets of functions ϕ that leave x fixed do form a group. The calculations of jets can conveniently be described in terms of jet groupoids.

We give a brief introduction to groupoids and jet groupoids. A full introduction can be found in Moerdijk and Mrčun [54], Mackenzie [50] or Pommaret [62].

Definition 4.8.1. A *groupoid* is a small category in which every morphism is invertible. $\ominus$

More concrete: a groupoid consists of a set of *objects* $X = G^{(0)}$, called the *base space*, and a set of *morphisms* or *arrows* $G = G^{(1)} \rightrightarrows X$, called the *total space*. We also have the source projection $s : G \to X$ and target projection $t : G \to X$. We can make the composition of two morphisms $g, h \in G$ if $t(h) = s(h)$. This composition defines a multiplication map $G \times_X G \to G$. The condition that every morphism is invertible implies that for every morphism g from x to y there is a unique inverse g^{-1} from y to x. For every point x in the base there is a unique unit morphism id_x from x to x. For all morphisms g with target x we have $\mathrm{id}_x \circ g = g$ and for all morphisms g with source x we have $g \circ \mathrm{id}_x = g$.

For every pair $x, y \in X$ we will denote by $G(x, y)$ the morphisms from x to y. By $G(x, -)$, $G(-, x)$ and $G_x = G(x, x)$ we denote the morphisms with $s(g) = x$, $t(g) = x$ and $s(g) = t(g) = x$, respectively. We also say they define the set of all morphisms to x, the morphisms from x and the morphisms over x, respectively. Since all morphisms are invertible it follows that $t(G(x, -)) = s(G(-, x))$. We let $Gx = t(G(x, -)) = s(G(-, x))$ be the orbit of base point x under the morphisms in G. We say a groupoid $G \rightrightarrows X$ is *transitive* if $Gx = X$.

Example 4.8.2 (Groupoid examples).

Pair groupoid Let X be any set. The *pair groupoid* of X is the groupoid of pairs of elements in X. The set X acts as the base space. The total space is given by $X \times X$ with s and t the natural projections onto the first and second component, respectively. Two morphisms (x, y) and (a, b) can be multiplied if and only if $y = a$ and in that case $(x, y) \cdot (a, b) = (x, b)$.

Equivalence groupoid Let X be a set and let $\sim$ be an equivalence relation on X. The *equivalence groupoid* of X is a subgroupoid of the pair groupoid of X. The set X is the base space and for every pair $x, y \in X$ there is a single morphism from x to y if and only if $x \sim y$.

Groups Every group G is a groupoid in a natural way. The base space consist of a single point e and the total space is equal to the group G. For a general groupoid $G \rightrightarrows X$ the morphisms $G(x, x)$ over a point x in the base form a group. ⊘

A *Lie groupoid* is a groupoid $G \rightrightarrows X$ for which G and X are smooth manifolds and all maps are smooth and the source and target maps are submersions. With these definitions it follows that G_x is a Lie group and $t : G(x, -) \to Gx$ is a principal G_x-bundle.

The inverse mapping theorem says that a diffeomorphism $\phi : M \to M$ is invertible near a point x if and only of the total derivative of ϕ at x is invertible. We say an r-jet j^r_x is invertible if the first order part is invertible.

Definition 4.8.3. Let X be smooth manifold. Define $\Pi_q = \Pi_q(X) = \Pi_q(X \times X) \subset \mathrm{J}^q(X, X)$ by the condition that the first order part of an element of $\mathrm{J}^q(X, X)$ is invertible. Then Π_q is a transitive Lie groupoid and is called the *jet groupoid* of X. The groupoid Π_q is the fiber bundle over $X \times X$ for which the fibers are invertible jets from x to y.

A *jet groupoid* $\mathcal{R}_q$ is a subset of Π_q that is closed with respect to all groupoid operations. A jet groupoid is also called a *system of Lie equations*. ⊖

Let $\mathcal{E} = X \times X$ and introduce coordinates (x, y) for $\mathcal{E}$. Then the variables x, y and $y^i_j = \partial y^i / \partial x^j$ form local coordinates for $\mathrm{J}^1(X, X)$. The jet groupoid is defined by $\det(y^i_j) \neq 0$. On every jet groupoid Π_q we have an identity section $\mathrm{id} : X \to \Pi_q$ given by the q-jet of the identity on X. We will write $\Pi_q(x, y)$ for the morphisms in $\Pi_q(X)$ with source x and target y.

Example 4.8.4 (Jet groupoid of $\mathbb{R}$). Let $X = \mathbb{R}$ and introduce coordinates (x, y) for $X \times X$. The 1-jet of a diffeomorphism $\phi : X \to X$ at x is equal to $(x, \phi(x), \phi'(x))$. The total space of the first order jet groupoid of X is isomorphic to $\mathbb{R}^2 \times \mathbb{R}^*$ and has coordinates (x, y, y_x). The source and target projections are the projection onto the first and second component, respectively. The composition of two 1-jets (x, y, y_x) and (y, z, z_x) is $(x, y, y_x) \cdot (y, z, z_y) = (x, z, y_x z_y)$. Inversion is given by $(x, y, y_x) \mapsto (y, x, 1/y_x)$. ⊘

Example 4.8.5 (Jet groupoid of linear transformations). The linear transformations of $X = \mathbb{R}$ are the maps $x \mapsto y = ax$. The 1-jets (x, y, y_x) of linear transformations all satisfy the condition $xy_x - y$. Since the composition of two linear transformations is again a linear transformation we expect the set of all 1-jets of invertible linear transformations to define a jet groupoid.

Suppose (x, y, y_x) and (y, z, z_x) are two 1-jets of invertible linear transformations and let $(x, z, z_x) = (x, y, y_x)\cdot(y, z, z_x)$. The composition of these two jets is given in Example 4.8.4. Then

$$xz_x - z = xz_y y_x - yz_y = z_y(xy_x - y) = 0.$$

Also $y(1/y_x) - x = (1/y_x)(y - xy_x) = 0$. Hence the subset of $\Pi_1(X)$ defined by

$$\mathcal{R} = \{\, (x, y, y_x) \in \Pi_1(X) \mid xy_x - y = 0 \,\}$$

is closed under the groupoid operations. Hence $\mathcal{R}$ is a jet groupoid. $\oslash$

The gauge groupoid construction. Every transitive Lie groupoids is equivalent to a principal bundle. See for example Moerdijk and Mrčun [54, pp. 114–115]. Let $P \to M$ be a principal bundle with structure group H. The structure group acts on the direct product bundle $P \times P$ and we can form the associated fiber bundle $\mathrm{Gauge}(P) = P \times_H P$. This fiber bundle is a groupoid. The morphisms are the orbits of the diagonal action of H on $P \times P$. The source and target projection are given by projection onto the first and second component of $P \times P$ followed by projection with π, respectively. The composition is defined such that $P \times P \to P \times_H P$ is a groupoid homomorphism from the pair groupoid to the gauge groupoid. The bundle $\mathrm{Gauge}(P)$ is called the *gauge groupoid* or Ehresmann groupoid of the principal bundle P. Conversely, let $G \rightrightarrows X$ be a transitive groupoid. The sequence $G_x \to G(x, -) \to X$ defines a principal G_x-bundle. The gauge groupoid of this principal bundle is isomorphic to $G \rightrightarrows X$.

Example 4.8.6. Let X be a smooth manifold and $\mathrm{F}X$ the frame bundle of X. The frame bundle is a principal bundle with structure group $G = \mathrm{GL}(V)$. Then the gauge groupoid $\mathrm{F}X \times_G \mathrm{F}X$ is isomorphic to $\Pi_1(X)$. Suppose (x, u), (x', u') are points in $\mathrm{F}X$ with $x, x' \in X$ and u, u' linear maps $T_xX \to V$ and $T_{x'}X \to V$, respectively. Then the map $\mathrm{F}X \times \mathrm{F}X \to \Pi_1(X) : (x, u), (x', u') \mapsto (x, x', (u')^{-1} \circ u)$ factorizes through the projection $\mathrm{F}X \times \mathrm{F}X \to \mathrm{F}X \times_G \mathrm{F}X$ and the isomorphism $\mathrm{F}X \times_G \mathrm{F}X \to \Pi_1(X)$ $\oslash$

Geometric structures. Many geometric structures can be given as reductions of the frame bundle. For example in Riemannian geometry a reduction of the frame bundle $\mathrm{F}X$ to an $\mathrm{O}(n)$-bundle defines a metric on X. Via the gauge groupoid construction this corresponds to the following. If we choose a frame for T_xX and we use the frame to identify T_xX with $\mathbb{R}^n$, then the group $\mathrm{O}(n)$ acts on T_xX. The action of $\mathrm{O}(n)$ on T_xX induces an action on $\Pi_1(x, x)$. We define the natural bundle by

$$\mathcal{F} = \Pi_1(-, x)/\,\mathrm{O}(n).$$

The group $\Pi_1(x, x)$ acts on $\mathcal{F}$ from the left. The sections of the natural bundle $\mathcal{F} \to X$ correspond to Riemannian metrics.

For a general jet groupoid $\mathcal{R}_q$ we define the *natural bundle* by

$$\mathcal{F} = \Pi_q(-, x)/\mathcal{R}_q(x, x).$$

The sections of the natural bundle define geometric structures of the same type as the original geometric structure. Other geometric structure with their corresponding structure groups are almost complex structures ($\mathrm{GL}(n, \mathbb{C})$), Kähler structures ($\mathrm{U}(n)$) and symplectic structures ($\mathrm{Sp}(2n, \mathbb{R})$).

Action of the diffeomorphism group. The diffeomorphisms of the base manifold X act on the geometric structure. This means that we can lift the action of any diffeomorphism to an action in the natural bundle. We are interested in the diffeomorphisms that leave invariant the geometric structure defined by the section of the natural bundle. If the structure is transitive we can restrict our attention to the diffeomorphism group fixing a base point y. The diffeomorphism group acts transitively on $\mathcal{F}_y$. The diffeomorphism group then acts by prolongation on the jet bundles $\mathrm{J}^r(\mathcal{F})$ of the natural bundle. If this action is not transitive at a certain order, then we have found differential invariants of the geometric structure. To calculate these differential invariants we note that the full diffeomorphism group acts on $\mathcal{F}_r$ only by a finite order part.

First order systems. The generalized first order order systems described in this chapter are defined by a rank 4 distribution on a 6-dimensional manifold, or equivalently as section of $\mathrm{Gr}_2(TM)$.

Let $\mathcal{V}$ be any rank four distribution on M. We define the jet groupoid $\mathcal{R}$ as the groupoid of jets $\{\, j^1_x\phi \in \Pi_1(M) \mid (T_x\phi)(\mathcal{V}_x) = \mathcal{V}_{\phi(x)} \,\}$. We define the group $G(4, 6)$ as the group of matrices

$$\begin{pmatrix} \mathrm{GL}(4, \mathbb{R}) & \mathrm{Lin}(2, 4) \\ 0 & \mathrm{GL}(2, \mathbb{R}) \end{pmatrix} \subset \mathrm{GL}(6, \mathbb{R}).$$

The group $G(4, 6)$ is a codimension 8 subgroup of $\mathrm{GL}(6, \mathbb{R})$. Choose a point $x \in M$ and an identification of T_xM with $\mathbb{R}^6$ such that $\mathcal{V}_x$ is spanned by the first four basis vectors in $\mathbb{R}^6$. With this identification $\mathcal{R}(x, x)$ is isomorphic to $G(4, 6)$. The natural bundle $\mathcal{F} = \Pi_1(-, x)/\mathcal{R}(x, x)$ has dimension 14 (the fibers of the bundle over M have dimension equal to 8). The natural bundle is diffeomorphic to $\mathrm{Gr}_2(TM)$. A section of the natural bundle corresponds to a rank 4 distribution on M.

The space of r-jets of rank 4 distributions on M equal to the prolongation $\mathrm{J}^r(\mathcal{F})$ of the natural bundle. Every local diffeomorphism of M acts on the r-jets of distributions. This gives an action of $\Pi_q(x, x)$ on $\mathrm{J}^{q-1}(\mathcal{F})_x$. The action of $\Pi_q(X)$ on $\mathrm{J}^0(\mathcal{F}) = \mathcal{F}$ is transitive, this follows from the fact that the group $\mathrm{GL}(6, \mathbb{R})$ acts transitively on the 4-dimensional linear subspaces of $\mathbb{R}^6$. Hence to study the invariants of the action it is sufficient to study the action of $\Pi_q(x, x)$ on $\mathrm{J}^{q-1}(\mathcal{F})_x$.

Order r	$\dim \mathrm{J}^r(\mathcal{F})_x$	$\dim \Pi_{r+1}(x, x)$	Orbit	Codimension
0	8	36	8	0
1	56	162	56	0
2	224	498	224	0
3	672	1254	670	2
4	1680	2766	1634	46

Table 4.1: Action of the diffeomorphism group on jets of distributions

The package JETS by Mohamed Barakat [7] has implemented routines to calculate the action of the diffeomorphism group on a geometric structure order by order. At each order q we calculate the dimension of the fibers of prolongation of the natural bundle $\mathrm{J}^r(\mathcal{F}) \to M$, the dimension of group $\Pi_q(x, x)$ acting and the maximal dimension of the orbits.

The codimension of the orbits with maximal dimension gives an upper bound for the number of (local) invariants for the rank 4 distributions on M. Near a generic orbit any transversal section locally defines invariants of the action. These invariants are only local and it can very well be that near non-generic orbits the invariants are not defined at all. The results of the calculation are in Table 4.1.

We see that there are no invariants at order 2. At order 3 there are at most 2 continuous invariants. These are invariants of the generic orbits. For non-generic orbits (these correspond to non-generic first order systems) these invariants might not be well-defined. At order 4 there are at most 44 additional invariants.

Remark 4.8.7. The generic first order systems are defined by generic rank 4 distributions in a 6-dimensional manifold. The second order equations are defined by rank 4 distributions in a 7-dimensional manifold. However, the distributions for second order equations are far from generic. This can clearly be seen in definition of a Vessiot system, Definition 4.1.1. The conditions for a rank 4 distribution on a 7-dimensional manifold to define a Vessiot system do not only depend on the value of the distribution $\mathcal{V}$ at a point, but on $\mathcal{V}'$ as well and hence on first order derivative of $\mathcal{V}'$. The same is true for the first order systems under point geometry (rank 4 distributions with an *integrable* rank 2 subdistribution in $\mathbb{R}^6$).

This implies that we cannot define the second order equations by arbitrary sections of the Grassmannian $\mathrm{Gr}_2(TM)$, M a manifold of dimension 7. We need to define the second order equations as a structure on a higher order bundle and this complicates the calculation of invariants. $\oslash$

4.8.2 Invariants at order 3

In the previous section we explained how we can calculate an upper bound on the number of invariants at each order. We concluded that the first invariants appear at order 3 and at this order there can be at most two functionally independent ones. In this section we make a geometric construction of these third order invariants.

We assume that we have a hyperbolic first order system that is generic in the sense that $\operatorname{rank} \mathcal{D} = 4$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \operatorname{span}(0)$. We use the genericity to construct an invariant coframing e_1, e_2, e_3 as described in Section 4.7.3. Then the distributions $\mathcal{B}_1$, TM_+ and TM_- are of order 2. We can define the distribution $\mathcal{W}_\pm = (\mathcal{B}_1)_\pm \oplus TM_\mp$ and this distribution is of order 2 as well. The Lie brackets modulo $\mathcal{W}_\pm$ define an invariant tensor $\mathcal{W}_\pm \times_M \mathcal{W}_\pm \to TM/\mathcal{W}_\pm$ of order 3. Most coefficients of this tensor have already been normalized, but not all of them. The Lie bracket of $(e_1)_+$ and $(e_3)_-$ modulo $\mathcal{W}_+$ is invariantly defined. Since e_1 and e_3 are of order 3, this expression is of order 3. We can write this as $[(e_1)_+, (e_3)_-] = (k_1)_+(e_2)_+ + (k_2)_+(e_3)_+ \mod \mathcal{W}_+$. The normalizations made so far imply $(k_2)_+ = 0$ but $(k_1)_+$ defines an invariant of order 3. For the other distribution $\mathcal{W}_-$ we can define $[(e_1)_-, (e_3)_+] = (k_1)_-(e_2)_- + (k_2)_-(e_3)_- \mod \mathcal{W}_-$. This gives a second invariant $(k_1)_-$.

At order 4 there are many invariants, and we make no attempt in classifying them. Since we have an invariant frame we have an invariant coframe as well. All the structure functions and coframe derivatives of this invariant coframe define invariants of the system.

Remark 4.8.8. In the next chapter we will develop a structure theory in differential forms for first order systems. In terms of an adapted coframing the invariants constructed are given by the coefficients S_{1F} after we have normalized $T_{2F} = U_{3F} = S_{3F} = 1$ and $T_{3F} = U_{2F} = V_{2F} = V_{3F} = S_{2F} = 0$. ⊘

Example 4.8.9 (Invariant framing). Consider the first order system

$$u_y = (u_x)^2 + (v_y)^2, \quad v_x = u.$$

We will use $x, y, u, v, p = u_x, s = v_y$ as coordinates on the equation manifold M. The calculations for this example can be found on the authors homepage [32]. Let $D_x = \partial_x + p\partial_u + u\partial_v$, $D_y = \partial_y + (p^2 + s^2)\partial_u + s\partial_v$. Take $H = \sqrt{2p^2 + 6s^2}/4$. On TM we define the framing

$$\begin{aligned}
e_1 &= H\partial_p, \\
e_2 &= -\frac{1}{32H^2}\left(D_y + pD_x + 2s(p^2+s^2)\partial_p + 4ps^2\partial_s\right), \\
e_3 &= -\frac{1}{16H}\left(D_x - \frac{ps(p^2-s^2)}{4H^2}\partial_p + (p^2+s^2)\partial_s\right), \\
e_4 &= \frac{H}{2}\partial_s, \\
e_5 &= (1/16)(\partial_u + 6s\partial_p + 2p\partial_s), \\
e_6 &= \frac{1}{64H}\left(\partial_v + 2s\partial_u + 16H^2\partial_p + \frac{ps(3p^2+5s^2)}{2H^2}\partial_s\right).
\end{aligned}$$

The contact distribution is $\mathcal{V} = \operatorname{span}(e_1, e_2, e_3, e_4)$ and the almost product structure on TM is given by $\operatorname{span}(e_1, e_3, e_5) \oplus \operatorname{span}(e_2, e_4, e_6)$. A calculation of the Nijenhuis tensor shows that $\mathcal{B}_1 = \mathrm{span}(e_1, e_2)$, $\mathcal{B}_2 = \mathrm{span}(e_3, e_4)$ and $\mathcal{B}_3 = \mathrm{span}(e_5, e_6)$. In the frame we have the following normalizations

$$\begin{aligned}
&[e_1, e_3] \equiv -e_5 \mod \mathcal{V},\\
&[e_2, e_4] \equiv e_6 \mod \mathcal{V},\\
&N(e_1, e_5) = e_4, \quad N(e_2, e_6) = e_3,\\
&N(e_3, e_5) = e_6, \quad N(e_4, e_6) = e_5.
\end{aligned}$$

Hence this framing is an invariant framing for a first order hyperbolic system as described in Section 4.7.3. The third order invariants are $(k_1)_+ = 0$ and $(k_1)_- = s(p^2 + 2s^2)/(16H^3)$.

Because the framing e_1, e_2, e_3 is invariant under contact transformations it follows from Theorem 1.2.58 that the contact symmetry group of the system is finite-dimensional. The dual coframe to the invariant framing is of rank one at generic points.

A calculation of this symmetry group using the MAPLE package JETS yields that the contact symmetry group of the system is 5-dimensional. The infinitesimal symmetries are given by

$$\begin{aligned}
&X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_v,\\
&X_4 = 2\partial_u + x\partial_v,\\
&X_5 = x\partial_x + 2y\partial_y + v\partial_v - p\partial_p - s\partial_s.
\end{aligned} \tag{4.24}$$

The symmetries X_1, X_2 and X_3 correspond to translations in the x, y and v directions. The symmetry X_4 is a combined translation in the direction u and a scaling of v by a factor $\exp(x)$. The last symmetry X_5 is a scaling symmetry. ⊘

Chapter 5

First order systems

In this chapter we develop the structure theory for first order systems in terms of differential forms. We also make connections to the theory of hyperbolic surfaces developed in Section 2.3.1 and theory developed by other mathematicians. For example hyperbolic exterior differential systems in Section 5.5.1 and linear partial differential operators in Section 5.5.2. We start with an analysis of the structure theory for hyperbolic systems. Then we review the elliptic theory and apply this to study base transformations.

The two different structure theories (distributions and differential forms) will be used together in Chapter 8 and Chapter 9. In these chapters we will need aspects from both theories to obtain our results.

5.1 Contact transformations and symmetries

An *external contact transformation* between two first order systems $M \subset \mathrm{Gr}_n(TB)$, $\tilde{M} \subset \mathrm{Gr}_n(T\tilde{B})$ is a local diffeomorphism ϕ from an open subset of $\mathrm{Gr}_n(TB)$ to some open subset of $\mathrm{Gr}_n(T\tilde{B})$ such that

- M is mapped to $\tilde{M}$
- The contact system is preserved, i.e., $\phi^*(\tilde{I}) \equiv 0 \mod I$. Another way of saying this is that the dual distributions should be mapped to each other: for all m in the domain of ϕ we should have $T_m\phi(\mathcal{V}_m) = \tilde{\mathcal{V}}_{\phi(m)}$.

An *internal contact transformation* is a (local) diffeomorphism $M \to \tilde{M}$ that preserves the pullback of the contact system of the Grassmannian. Another equivalent definition is that an internal contact transformation is local diffeomorphism of $\mathrm{Gr}_n(TB)$ to $\mathrm{Gr}_n(T\tilde{B})$ such that

- M is mapped to $\tilde{M}$
- The contact system is preserved on M, i.e., $\phi^*(\tilde{I})_m \equiv 0 \mod I_m$ for all points $m \in M$. In terms of the dual distributions this condition is that for all $m \in M$ with m in the domain of ϕ we should have $T_m\phi(\mathcal{V}_m) = \tilde{\mathcal{V}}_{\phi(m)}$.

In the following we will mean by a contact transformation always an internal contact transformation. Every external contact transformation restricts to an internal contact transformation, but the converse is not true.

Let $\pi : \mathrm{Gr}_n(TB) \to B$ and $\tilde{\pi} : \mathrm{Gr}_n(T\tilde{B}) \to \tilde{B}$ be the projections to the base manifolds. For every (local) diffeomorphism $\psi : B \to \tilde{B}$ of the base manifolds, there is a unique contact transformation $\Psi : \mathrm{Gr}_n(TB) \to \mathrm{Gr}_n(T\tilde{B})$ such that $\tilde{\pi} \circ \Psi = \psi \circ \pi$. The contact transformation Ψ is defined by $\Psi : L \in \mathrm{Gr}_2(T_bB) \mapsto T_b\psi(L) \in \mathrm{Gr}_2(T_{\psi(b)}\tilde{B})$. We call this the *lift* or *induced map* of ψ. This definition is very similar to the definition of the lift of an equivalence of G-structures described on page 13. The contact transformations that are lifts of diffeomorphisms of the base are called *point transformations*.

In the special situation that $B = \tilde{B}$ and $M = \tilde{M}$ we call an external contact transformation an *external symmetry* of M. An *internal symmetry* is an internal contact transformation from M to M. It is clear that any external symmetry restricts to an internal symmetry, but conversely not every internal symmetry extends to an external symmetry. For more information on internal and external symmetries see the paper by Anderson et al. [5].

We make an important observation for a codimension c system. Let $s = \dim B - n$. For the contact system on $M \subset \mathrm{Gr}_n(TB)$ for $s > 2$ the fibers of the projection $\mathrm{Gr}_2(TB) \to B$ are integral manifolds for which the tangent spaces are maximal integral elements of dimension $ns - c$. In contrast, the graphs of 1-jets of functions $u(x)$ define n-dimensional integral manifolds. The tangent spaces are n-dimensional maximal integral elements. Whenever $ns - c > n$ the fibers of the projection $M \to B$ are geometrically different from the generic integral manifolds. This implies the following theorem:

Theorem 5.1.1. *Let M be a codimension c system in* $\mathrm{Gr}_n(TB)$*. If $ns - c > n$, then all internal symmetries of the equation are external symmetries and are induced from base transformations.*

Proof. Any symmetry must map the fibers (which are integral manifolds of maximal dimension) to fibers and therefore induces a transformation on the base. It is easily checked that the symmetry defined by the prolongation of this base transformation must be equal to the original symmetry. □

For $c = 0$ this theorem is a special case of Bäcklund's theorem. Bäcklund's theorem deals with the external contact transformations.

The systems with $n = 1$ are ordinary differential equations. The first interesting systems of partial differential equations occur for $n = s = 2$. If $c = 1$, then there are no interesting internal symmetries. An analysis of the cases $c = 3, 4$ using the structure equations for the system is done in Bryant et al. [13, Chapter VII, §2]. In this dissertation we study the determined first order systems $n = s = c = 2$.

Example 5.1.2. Consider the first order jet bundle $M = \mathrm{J}^1(\mathbb{R}, \mathbb{R}^2)$. This jet bundle can be seen as an open subset of the Grassmannian $\mathrm{Gr}_1(TB)$ of 1-planes in $B = \mathbb{R} \times \mathbb{R}^2$. We take local coordinates x, u, v for the base manifold. We have coordinates $x, u, v, p = u_x, r = v_x$ for M. The contact system is generated by two 1-forms $\theta^1 = \mathrm{d}u - p\mathrm{d}x$, $\theta^2 = \mathrm{d}v - r\mathrm{d}x$. The dual distribution $\mathcal{V}$ has basis

$$X = \partial_x + p\partial_u + r\partial_v, \quad \partial_p, \quad \partial_r. \tag{5.1}$$

Every non-zero vector in $\mathcal{V}$ spans a 1-dimensional integral element, hence $V_1(\mathcal{V}) = \mathbb{P}\mathcal{V}$. At each point of M there is only one 2-dimensional integral element and it is spanned by the vectors ∂_p, ∂_r. The integral elements of maximal dimension (two in this example) form an integrable distribution. The leaves of this distribution are precisely the fibers of the projection $\mathrm{Gr}_1(TB) \to B$. ⊘

5.2 Hyperbolic structure theory

In this section we will develop the structure theory for first order hyperbolic systems. The theory will use differential forms as the main objects, in contrast with the structure theory using distributions developed in the previous chapter.

Many classical mathematicians (Lie, Vessiot, Cartan) studied equations of hyperbolic type. The equations of elliptic type were assumed to be analytic and then treated in complex coordinates (where everything is hyperbolic). This complexification is also used in Stormark [64, pp. 279–280].

Some recent authors [4, 38, 44, 66] deal only with hyperbolic systems. In contrast the work of McKay [51, 52] only deals with the elliptic case. McKay believed the elliptic case is simpler because we can use the familiar concepts of complex structures. Below we will see that using hyperbolic variables and almost product structures, the structure theory introduced by McKay is almost identical in the hyperbolic and elliptic cases.

5.2.1 Point geometry

In this section we develop a structure theory for first order systems using differential forms and structure equations. We will rediscover the almost product structure and the Nijenhuis tensor as invariant structures. The starting point is a base manifold of dimension four and a codimension two submanifold M in $\mathrm{Gr}_2(TB)$ that is transversal to the projection $\mathrm{Gr}_2(TB) \to B$. The intersection of the tangent space to the fibers of the projection and the tangent space to M defines a rank two integrable distribution. The leaves of this distribution define the *base foliation*. The contact system on $\mathrm{Gr}_2(TB)$ pulls back to a contact system on M. On M we can choose 1-forms θ^1, θ^2 that form a basis for the contact system. We can then choose two additional 1-forms ω^1, ω^2 such that the distribution defining the base foliation is equal to the dual of $\mathrm{span}(\theta^1, \theta^2, \omega^1, \omega^2)$. We complete the coframe by choosing two additional 1-forms π^1, π^2.

Recall that FM is the bundle of frames (or coframes) on the manifold B. The coframe θ^1, $\theta^2, \omega^1, \omega^2, \pi^1, \pi^2$ constructed above defines a special section of the bundle of frames. In this section we will start with the subbundle of FM that contains all coframes $\theta^1, \theta^2, \omega^1, \omega^2, \pi^1$, π^2 such that $\mathrm{span}(\theta^1, \theta^2)$ is the contact system and $\mathrm{span}(\theta^1, \theta^2, \omega^1, \omega^2)^\perp$ is the distribution given by the tangent space of the fibers of $M \to B$. We will then reduce this bundle step by step to bundles for which the sections are coframes on M that are adapted to the geometry on M. The final result is Definition 5.2.3 in which the bundle of adapted coframes is defined.

For convenience we will assume that on $(M, \mathcal{V})$ we have made a choice of positive and negative Monge system. This will reduce the structure group by a discrete factor and this will make the notation much easier. In local coordinates the independence condition for solutions to the systems will be given by $\omega^1 \wedge \omega^2 \neq 0$.

Remark 5.2.1. The constructions in this section are similar to the sections 6.2 – 6.4 in McKay [51] where a structure theory for elliptic systems is developed. For elliptic systems it turned out to be useful to write everything in complex coordinates. For hyperbolic systems we use hyperbolic numbers to express the structure equations. Note that in McKay [51] the action of the structure group on A on top of page 39 and Proposition 12 are incorrect, since e should be complex-linear as well. The final results of McKay in that section are correct. $\oslash$

We will write θ for the $\mathbb{D}$-valued differential form $(\theta^1, \theta^2)^T$ and similarly for ω and π. From the definition of a first order system it follows that the structure equations are given by

$$\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = -\begin{pmatrix}\tilde{\alpha} & 0 & 0\\ \tilde{\beta} & \tilde{\gamma} & 0\\ \tilde{\delta} & \tilde{\epsilon} & \tilde{\zeta}\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} - \begin{pmatrix}\tilde{A}\wedge\omega\\ 0\\ 0\end{pmatrix}$$

for 2×2-matrices $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$, $\tilde{\epsilon}$, $\tilde{A}$. The structure group for the coframe is the group of matrices

$$\begin{pmatrix}\tilde{a} & 0 & 0\\ \tilde{b} & \tilde{c} & 0\\ \tilde{d} & \tilde{e} & \tilde{f}\end{pmatrix} \in \mathrm{GL}(6, \mathbb{R}).$$

Modulo $I = \mathrm{span}(\theta^1, \theta^2)$ we have $\mathrm{d}\theta^i \equiv -A^i_{jk}\pi^j \wedge \omega^k$. We write $\tilde{A}$ for the matrix $(\tilde{A})^i_k = A^i_{jk}\pi^j$. The structure group acts on the matrix $\tilde{A}$ as $\tilde{A} \mapsto \tilde{a}\tilde{A}\tilde{c}^{-1}$. This is a conformal action, see Section A.5. Using a transformation of the coframing we can arrange that in fact (this follows from the assumption the system is hyperbolic)

$$\mathrm{d}\theta \equiv -\begin{pmatrix}\pi^1 \wedge \omega^1\\ \pi^2 \wedge \omega^2\end{pmatrix} \equiv -\pi \wedge \omega \mod I. \tag{5.2}$$

This normalization reduces the structure group. In the reduced structure group $a'' = c'' = e'' = 0$ and $f = ac^{-1}$. The diagonal matrix $\mathrm{diag}(L, L, L)$, with L given in (2.5), does preserve the normalization (5.2) and interchanges the two Monge systems. Our choice of a positive Monge system eliminates the component of the structure group that includes the matrix $\mathrm{diag}(L, L, L)$.

We arrive at a coframing with structure equations

$$
\mathrm{d}\begin{pmatrix}\theta^1\\ \theta^2\\ \omega^1\\ \omega^2\\ \pi^1\\ \pi^2\end{pmatrix} = -\begin{pmatrix}\alpha_1 & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha_2 & 0 & 0 & 0 & 0\\ \beta_1 & \beta_I & \gamma_1 & 0 & 0 & 0\\ \beta_{II} & \beta_2 & 0 & \gamma_2 & 0 & 0\\ \delta_1 & \delta_I & \epsilon_1 & 0 & \alpha_1-\gamma_1 & 0\\ \delta_{II} & \delta_2 & 0 & \epsilon_2 & 0 & \alpha_2-\gamma_2\end{pmatrix} \wedge \begin{pmatrix}\theta^1\\ \theta^2\\ \omega^1\\ \omega^2\\ \pi^1\\ \pi^2\end{pmatrix}
- \begin{pmatrix}\pi^1\wedge\omega^1\\ \pi^2\wedge\omega^2\\ 0\\ 0\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\xi^1\wedge\theta^2\\ \xi^2\wedge\theta^1\\ \eta^1\wedge\omega^2\\ \eta^2\wedge\omega^1\\ \epsilon_I\wedge\omega^2+E_1\pi^1\wedge\pi^2\\ \epsilon_{II}\wedge\omega^1+E_2\pi^2\wedge\pi^1\end{pmatrix}
$$

and structure group given by matrices in GL(6, $\mathbb{R}$) of the form

$$
\begin{pmatrix}a & 0 & 0\\ \tilde{b} & c & 0\\ \tilde{d} & e & ac^{-1}\end{pmatrix} = \begin{pmatrix}a_1 & 0 & 0 & 0 & 0 & 0\\ 0 & a_2 & 0 & 0 & 0 & 0\\ b_1 & b_I & c_1 & 0 & 0 & 0\\ b_{II} & b_2 & 0 & c_2 & 0 & 0\\ d_1 & d_I & e_1 & 0 & a_1c_1^{-1} & 0\\ d_{II} & d_2 & 0 & e_2 & 0 & a_2c_2^{-1}\end{pmatrix}. \tag{5.3}
$$

In the notation with hyperbolic numbers introduced above the structure equations are given in the more compact form

$$
\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = \begin{pmatrix}\alpha & 0 & 0\\ \tilde{\beta} & \gamma & 0\\ \tilde{\delta} & \epsilon & \alpha-\gamma\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} - \begin{pmatrix}\pi\wedge\omega\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\xi\wedge\theta^F\\ \eta\wedge\omega^F\\ \epsilon''\wedge\omega^F+E\pi\wedge\pi^F\end{pmatrix},
$$

where α, γ, ϵ are diagonal matrices. For convenience we have written α instead of α', and similar for γ and ϵ. We can absorb terms in $\alpha, \tilde{\beta}, \gamma, \tilde{\delta}, \tilde{\epsilon}$ such that $\xi = \tilde{A}_2\omega + \tilde{A}_3\pi$ and $\eta = \tilde{B}_3\pi$ for 2×2-matrices $\tilde{A}_2, \tilde{A}_3, \tilde{B}_3$. We analyze the structure equations by calculating the consequences of $\mathrm{d}^2\theta = 0$ modulo θ, θ^F. We find

$$
\begin{aligned}
0 = \mathrm{d}^2\theta &\equiv \alpha\wedge\mathrm{d}\theta - \mathrm{d}\pi\wedge\omega + \pi\wedge\mathrm{d}\omega - \xi\wedge d\theta^F\\
&\equiv -\alpha\wedge\pi\wedge\omega + (\alpha-\gamma)\wedge\pi\wedge\omega + \epsilon''\wedge\omega^F\wedge\omega\\
&\quad - E\pi\wedge\pi^F\wedge\omega - \pi\wedge\gamma\wedge\omega\\
&\quad + \pi\wedge\eta\wedge\omega^F + \xi\wedge\pi^F\wedge\omega^F\\
&\equiv (-E\omega + B_3{}''\omega^F + A_3{}'\omega^F)\wedge\pi\wedge\pi^F + (-\epsilon'' - A_2{}'\pi^F)\wedge\omega\wedge\omega^F.
\end{aligned}
$$

From this we can conclude that $E = 0$, $A_3{}' + B_3{}'' = 0$ and $(\epsilon'' + A_2{}'\pi^F)\wedge\theta\wedge\theta^F\wedge\omega\wedge\omega^F = 0$. Since we can absorb terms θ, ω from ϵ'' into $\tilde{\delta}$ and ϵ' we can arrange that $\epsilon'' = -A_2{}'\pi^F$.

The new structure equations are

$$\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = -\begin{pmatrix}\alpha & 0 & 0\\ \tilde{\beta} & \gamma & 0\\ \tilde{\delta} & \epsilon & \alpha-\gamma\end{pmatrix}\wedge\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} - \begin{pmatrix}\pi\wedge\omega\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\xi\wedge\theta^F\\ \eta\wedge\omega^F\\ -A_2{}'\pi^F\wedge\omega^F\end{pmatrix},$$

with $A_2{}' \in \mathbb{D}$. We again calculate $\mathrm{d}^2\theta$, but this time only modulo θ.

$$\begin{aligned}
0 = \mathrm{d}^2\theta &\equiv \alpha\wedge \mathrm{d}\theta - \mathrm{d}\pi\wedge\omega + \pi\wedge \mathrm{d}\omega\\
&\quad + \mathrm{d}(\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge\theta^F - (\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge \mathrm{d}\theta^F\\
&\equiv -\alpha\wedge\pi\wedge\omega + \alpha\wedge(\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge\theta^F + \delta''\wedge\theta^F\wedge\omega\\
&\quad + (\alpha-\gamma)\wedge\pi\wedge\omega + A_2{}'\pi^F\wedge\omega^F\wedge\omega - \pi\wedge\beta''\wedge\theta^F\\
&\quad - \pi\wedge\gamma\wedge\omega + \pi\wedge\eta\wedge\omega^F + \mathrm{d}(\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge\theta^F\\
&\quad + (\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge\alpha^F\wedge\theta^F + (\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge\pi^F\wedge\omega^F\\
&\equiv \alpha\wedge(\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge\theta^F + \delta''\wedge\theta^F\wedge\omega + A_2{}'\pi^F\wedge\omega^F\wedge\omega\\
&\quad - \pi\wedge\beta''\wedge\theta^F + \pi\wedge(-A_3{}'\pi^F)\wedge\omega^F + \mathrm{d}(\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge\theta^F\\
&\quad - (\tilde{A}_2\omega + \tilde{A}_3\pi)\wedge\alpha^F\wedge\theta^F + (A_2{}'\omega + A_3{}'\pi)\wedge\pi^F\wedge\omega^F\\
&\equiv \big[\mathrm{d}(\tilde{A}_2\omega + \tilde{A}_3\pi) - \delta''\wedge\omega + \beta''\wedge\pi + \alpha\wedge(\tilde{A}_2\omega + \tilde{A}_3\pi)\\
&\quad + \alpha^F\wedge(\tilde{A}_2\omega + \tilde{A}_3\pi)\big]\wedge\theta^F\\
&\equiv (\mathrm{d}A_2{}' - \delta'' + A_2{}'\alpha - A_2{}'\alpha^F)\wedge\omega\wedge\theta^F\\
&\quad + (\mathrm{d}A_2{}'' + A_2{}''\alpha - A_2{}''\alpha^F)\wedge\omega^F\wedge\theta^F\\
&\quad + (\mathrm{d}A_3{}' + \beta'' + A_3{}'\alpha - A_3{}'\alpha^F)\wedge\pi\wedge\theta^F\\
&\quad + (\mathrm{d}A_3{}'' + A_3{}''\alpha - A_3{}''\alpha^F)\wedge\pi^F\wedge\theta^F\\
&\quad + (A_2{}'\mathrm{d}\omega + A_2{}''\mathrm{d}\omega^F + A_3{}'\mathrm{d}\pi + A_3{}''\mathrm{d}\pi^F)\wedge\theta^F \quad \mathrm{mod}\ \theta.
\end{aligned}$$

We introduce the forms

$$\begin{aligned}
\nabla A_2{}' &= \mathrm{d}A_2{}' - \delta'' + A_2{}'(\alpha - \alpha^F - \gamma) - A_3\epsilon,\\
\nabla A_2{}'' &= \mathrm{d}A_2{}'' + A_2{}''(\alpha - \alpha^F - \gamma^F) - A_3{}''\epsilon^F,\\
\nabla A_3{}' &= \mathrm{d}A_3{}' + \beta'' + A_3{}'(\gamma - \alpha^F),\\
\nabla A_3{}'' &= \mathrm{d}A_3{}'' + A_3{}''(\alpha - 2\alpha^F + \gamma^F).
\end{aligned} \tag{5.4}$$

Remark 5.2.2. The notation ∇ is inspired by the fact that the forms ∇A_j are in fact covariant derivatives with respect to the connection chosen. See McKay [53]. $\oslash$

With this we have

$$
\begin{aligned}
0 = \mathrm{d}^2\theta \equiv (\nabla A_2{}' \wedge \omega + \nabla A_2{}'' \wedge \omega^F + \nabla A_3{}' \wedge \pi + \nabla A_3{}'' \wedge \pi^F) \wedge \theta^F \\
+ (A_2{}'\tilde{B}_3\pi \wedge \omega^F + A_2{}''(\tilde{B}_3)^F\pi^F \wedge \omega \\
- A_2{}''A_3{}''\pi^F \wedge \omega^F + A_3{}''(A_2{}')^F\pi \wedge \omega) \wedge \theta^F \quad \mathrm{mod}\ \theta.
\end{aligned} \tag{5.5}
$$

This proves that the 1-forms ∇A_* are semi-basic with respect to θ, θ^F, ω, ω^F, π, π^F. The forms α, $\tilde{\beta}$, γ, $\tilde{\delta}$, ϵ define a connection and to this connection we can associate a covariant derivative acting on equivariant tensors on the bundle. The construction of a covariant derivative is equivalent to the covariant derivative that is constructed when we have a G-equivariant connection on a principal bundle. The construction of covariant derivatives in the context of connections that are not G-equivariant is described briefly in McKay [53]. The torsion A satisfies

$$
\mathrm{d}A = \nabla A + \rho(\alpha, \ldots, \epsilon)A,
$$

where ∇A is the covariant derivative and ρ is an affine representation of the Lie algebra on the torsion bundle. Compare this to (5.4).

When moving in a fiber of the bundle of adapted coframes over M we have $\nabla A = 0$, since ∇A is semi-basic. Therefore $\mathrm{d}A$ is determined by the representation ρ. The terms β'' and δ'' in (5.4) ensure that if we move in the direction of the vectors dual to β'', only $A_2{}'$ will change, so we can transform $A_2{}'$ into zero. Similarly for δ'' and $A_3{}'$. From the calculations above it follows that we can restrict to a subbundle on which $A_2{}' = A_3{}' = 0$. The reduced structure group is the subgroup of the previous structure group (5.3) for which $b'' = d'' = 0$. If at the same time we rearrange the terms involving β'', δ'', then we find the structure equations below.

We rename the terms $T_{2^F} = A_2{}''$, $T_{3^F} = A_3{}''$, and $S_{2^F} = -B_3{}'$. At the same time we have new terms $U_2, U_{2^F}, U_{3^F}, V_2, V_{2^F}, V_{3^F}, S_{1^F} \in \mathbb{D}$ because the structure group is reduced. With this new notation the structure equations become

$$
\begin{aligned}
\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = -\begin{pmatrix}\alpha & 0 & 0\\ \beta & \gamma & 0\\ \delta & \epsilon & \alpha-\gamma\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} - \begin{pmatrix}\pi \wedge \omega\\ \pi \wedge (S_{1^F}\theta^F + S_{2^F}\omega^F)\\ 0\end{pmatrix} \\
+ \begin{pmatrix} T_{2^F}\omega^F \wedge \theta^F + T_{3^F}\pi^F \wedge \theta^F \\ U_2\omega \wedge \theta^F + U_{2^F}\omega^F \wedge \theta^F + U_{3^F}\pi^F \wedge \theta^F \\ V_3\pi \wedge \theta^F + V_{2^F}\omega^F \wedge \theta^F + V_{3^F}\pi^F \wedge \theta^F \end{pmatrix}.
\end{aligned}
$$

We can absorb the term $U_2\omega \wedge \theta^F$ into γ. A calculation of $\mathrm{d}^2\theta$ modulo θ, ω^F, π^F yields that $V_3 = -U_2$ and hence V_3 and U_2 are absorbed at the same time.

We have constructed an adapted coframing by normalization of the structure functions. Without making any further assumptions on the structure functions we cannot make any further normalizations.

Definition 5.2.3 (Adapted coframing for hyperbolic first order system). Let θ, ω, π be a coframing on a hyperbolic first order system under point geometry. We say the coframing is adapted if it satisfies the structure equations

$$\begin{aligned} \mathrm{d}\theta = &-\begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ \delta & \epsilon & \alpha-\gamma \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} + \begin{pmatrix} -\pi\wedge\omega \\ -\pi\wedge\sigma \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} T_{2^F}\omega^F \wedge \theta^F + T_{3^F}\pi^F \wedge \theta^F \\ U_{2^F}\omega^F \wedge \theta^F + U_{3^F}\pi^F \wedge \theta^F \\ V_{2^F}\omega^F \wedge \theta^F + V_{3^F}\pi^F \wedge \theta^F \end{pmatrix}. \end{aligned} \tag{5.6}$$

Here $\sigma = S_{1^F}\theta^F + S_{2^F}\omega^F$. The bundle of all adapted coframes is written as B_M. The calculations above show that for any hyperbolic first order system the bundle B_M is a reduction of FM with structure group H. The group H is the group of invertible matrices

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix} \in \mathrm{GL}(6, \mathbb{R}), \tag{5.7}$$

with $a, b, c, d, e \in \mathbb{D}$. ⊖

For future reference we write down the covariant derivatives of all the torsion functions.

$$\begin{aligned} \nabla T_{2^F} &= \mathrm{d}T_{2^F} + T_{2^F}(\alpha - \alpha^F - \gamma^F) - T_{3^F}\epsilon^F, \\ \nabla T_{3^F} &= \mathrm{d}T_{3^F} + T_{3^F}(\alpha - 2\alpha^F + \gamma^F), \\ \nabla S_{1^F} &= \mathrm{d}S_{1^F} - \beta^F S_{2^F} + S_{1^F}(2\gamma - \alpha - \alpha^F), \\ \nabla S_{2^F} &= \mathrm{d}S_{2^F} + S_{2^F}(2\gamma - \alpha - \gamma^F), \\ \nabla U_{2^F} &= \mathrm{d}U_{2^F} + \beta U_{2^F} + U_{2^F}(\gamma - \gamma^F - \alpha^F) - U_{3^F}\epsilon^F, \\ \nabla U_{3^F} &= \mathrm{d}U_{3^F} + \beta U_{3^F} + U_{3^F}(\gamma + \gamma^F - 2\alpha^F), \\ \nabla V_{2^F} &= \mathrm{d}V_{2^F} + \delta V_{2^F} + \epsilon U_2 + V_{2^F}(\alpha - \alpha^F - \gamma - \gamma^F) - V_{3^F}\epsilon^F, \\ \nabla V_{3^F} &= \mathrm{d}V_{3^F} + \delta V_3 + \epsilon U_3 + V_{3^F}(\alpha - 2\alpha^F + \gamma^F - \gamma). \end{aligned} \tag{5.8}$$

The covariant derivatives above give the infinitesimal action of the structure group on the invariants. We will also calculate this action directly.

Action of the structure group. Let G be the structure group preserving this adapted coframing, i.e., G is the group of lower triangular matrices in $\mathrm{GL}(3, \mathbb{D})$ with diagonal entries (a, c, ac^{-1}). We want to analyze the action of this structure group on the invariants $S =$

(S_{1^F}, S_{2^F}), $T = (T_{2^F}, T_{3^F})$, U and V. Let

$$
\begin{aligned}
g &= \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix}, \\
g^{-1} &= \begin{pmatrix} a^{-1} & 0 & 0 \\ -ba^{-1}c^{-1} & c^{-1} & 0 \\ a^{-2}be - a^{-2}cd & -a^{-1}e & a^{-1}c \end{pmatrix} = \begin{pmatrix} \hat{a} & 0 & 0 \\ \hat{b} & \hat{c} & 0 \\ \hat{d} & \hat{e} & \hat{a}\hat{c}^{-1} \end{pmatrix}
\end{aligned}
\tag{5.9}
$$

and define $(\tilde{\theta}, \tilde{\omega}, \tilde{\pi})^T = g^{-1}(\theta, \omega, \pi)^T$. The new structure equations for $\tilde{\theta}$ become

$$
\begin{aligned}
\mathrm{d}\tilde{\theta} = \mathrm{d}(a^{-1}\theta) &\equiv -a^{-1}(\mathrm{d}a)a^{-1} \wedge \theta + a^{-1}\mathrm{d}\theta \\
&\equiv a^{-1}(-\pi \wedge \omega + \tau_1 \wedge \theta^F) \\
&\equiv -\tilde{\pi} \wedge \tilde{\omega} + a^{-1}\tau_1 \wedge \theta^F \quad \bmod \tilde{\theta}.
\end{aligned}
$$

The last term transforms as

$$
\begin{aligned}
a^{-1}\tau_1 \wedge \theta^F &= a^{-1}(T_{2^F}, T_{3^F}) \wedge (\omega^F, \pi^F)^T \wedge a^F \bar{\tilde{\theta}} \\
&= a^{-1}(T_{2^F}, T_{3^F}) \wedge \begin{pmatrix} c & 0 \\ e & ac^{-1} \end{pmatrix}^F (\tilde{\omega}^F, \tilde{\pi}^F)^T \wedge \bar{\tilde{\theta}} \\
&= a^{-1}(\widetilde{T}_{2^F}, \widetilde{T}_{3^F}) \wedge (\tilde{\omega}^F, \tilde{\pi}^F)^T \wedge \tilde{\theta}^F \\
&= \tilde{\tau}_1 \wedge \tilde{\theta}^F.
\end{aligned}
$$

Combining this we see the invariants transform as

$$
\begin{pmatrix} \widetilde{T}_{2^F} \\ \widetilde{T}_{3^F} \end{pmatrix} = a^{-1}a^F \begin{pmatrix} c^F & e^F \\ 0 & a^F(c^F)^{-1} \end{pmatrix} \begin{pmatrix} T_{2^F} \\ T_{3^F} \end{pmatrix}.
$$

The calculations for ω and π are more involved, but since $\mathrm{d}\tilde{\Phi} = \mathrm{d}(g^{-1}\Phi) = -g^{-1}(\mathrm{d}g) \wedge g^{-1}\Phi + g^{-1}\mathrm{d}\Phi$ and we are interested in the action of G on the invariants S, T, U, V (and not on the connection forms $\alpha, \beta, \ldots, \epsilon$) we can restrict ourselves to the action caused on the torsion component the structure equations, i.e., we only calculate the action of G on the torsion bundle. We have

$$
\begin{aligned}
\mathrm{d}\tilde{\omega} &\equiv \hat{b}\mathrm{d}\theta + \hat{c}\mathrm{d}\omega \\
&\equiv \hat{b}(-\alpha \wedge \theta - \pi \wedge \omega + \tau_1 \wedge \theta^F) + \hat{c}(-\beta \wedge \theta - \gamma \wedge \omega - \pi \wedge \sigma + \tau_2 \wedge \theta^F) \\
&\equiv \hat{b}\tau_1 \wedge \theta^F - \hat{c}\pi \wedge \sigma + \hat{c}\tau_2 \wedge \theta^F \\
&\equiv -\hat{c}ac^{-1}\tilde{\pi} \wedge \sigma + \hat{b}\tau_1 \wedge \theta^F + \hat{c}\tau_2 \wedge \theta^F \quad \bmod \tilde{\theta}, \tilde{\omega}.
\end{aligned}
$$

The terms $\tau_1 \wedge \theta^F$ and $\tau_2 \wedge \theta^F$ transform in precisely the same way as the term $\tau_1 \wedge \theta^F$ we already encountered when calculating $\mathrm{d}\tilde{\theta}$.

Define the following representations of the structure group on $\mathbb{D}^2$

$$\pi_\tau : G \to \mathrm{Aut}(\mathbb{D}^2) : g \mapsto a^F \begin{pmatrix} c^F & e^F \\ 0 & a^F (c^F)^{-1} \end{pmatrix},$$
$$\pi_\sigma : G \to \mathrm{Aut}(\mathbb{D}^2) : g \mapsto ac^{-1} \begin{pmatrix} a^F & b^F \\ 0 & c^F \end{pmatrix}.$$

It is not difficult to check that under the action of G the invariants $S = (S_{1^F}, S_{2^F})$ transform under the representation $\hat{c}\pi_\sigma$ and the invariants T, U, V transform as

$$\begin{aligned}
\tilde{T} &= a^{-1}\pi_\tau T, \\
\tilde{U} &= \hat{b}\pi_\tau T + \hat{c}\pi_\tau U, \\
\tilde{V} &= \hat{d}\pi_\tau T + \hat{e}\pi_\tau U + a^{-1}c\pi_\tau V.
\end{aligned}$$

The invariants T_{3^F} and S_{2^F} transform as

$$T_{3^F} \mapsto a^{-1}(a^F)^2(c^F)^{-1}T_{3^F}, \quad S_{2^F} \mapsto ac^{-2}c^F S_{2^F}.$$

Hence both T_{3^F} and S_{2^F} are relative invariants. By a *relative invariant* we mean that the zero set $T_{3^F} = 0$ is invariant under the structure group.

The structure group has dimension 10 and has an affine representation on the 16-dimensional linear space with coordinates $T_{2^F}, T_{3^F}, U_{2^F}, U_{3^F}, V_{2^F}, V_{3^F}, S_{1^F}, S_{2^F} \in \mathbb{D}$. Hence for generic first order systems we expect 6 real invariants at this order. One of the invariants is $|T_{3^F}|^2/|S_{2^F}|^2$, if this quotient is well-defined. At points where $S_{2^F} \in \mathbb{D}^*$, $T_{3^F} \in \mathbb{D}^*$ the orbits of the structure group have dimension 9 or 10, depending on the values of V_{2^F} and U_{2^F}.

A possible normalization scheme in the generic situation: use b to transform $S_{1^F} = 0$, use e to transform $T_{2^F} = 0$, use d to transform $V_{3^F} = 0$. The pair a, c acts on T_{3^F}, S_{2^F} with 3-dimensional orbits. The remaining freedom after normalization is $(a, c) \mapsto (\phi a, \phi c)$. Then the last degree of freedom ϕ acts on $U_{2^F}, U_{3^F}, V_{2^F}$.

Example 5.2.4. We consider the system $q = r$, $q = \phi(s)$. This first order systems can be obtained from the second order equation $z_{xy} = \phi(z_{yy})$ by taking the quotient by the symmetry ∂_z (see Chapter 9). By calculating $\mathrm{d}\theta$ for an initial coframing θ, ω, π we can find by some trial and error the following adapted coframing:

$$\begin{aligned}
\theta^1 &= (\mathrm{d}u - p\mathrm{d}x - r\mathrm{d}y) - \phi'(\mathrm{d}v - q\mathrm{d}x - s\mathrm{d}y), \\
\theta^2 &= \mathrm{d}v - q\mathrm{d}x - s\mathrm{d}y, \\
\omega^1 &= \mathrm{d}x, \quad \omega^2 = \mathrm{d}y + \phi'\mathrm{d}x, \\
\pi^1 &= \mathrm{d}p - \phi''\mathrm{d}s, \quad \pi^2 = \mathrm{d}s.
\end{aligned}$$

The structure equations are

$$
\begin{aligned}
\mathrm{d}\theta^1 &= -\pi^1 \wedge \omega^1 - \phi''\pi^2 \wedge \theta^2, \\
\mathrm{d}\theta^2 &= -\pi^2 \wedge \omega^2, \\
\mathrm{d}\omega^1 &= 0, \quad \mathrm{d}\omega^2 = \phi''\pi^2 \wedge \omega^1, \\
\mathrm{d}\pi^1 &= \mathrm{d}p - \phi''\mathrm{d}s, \quad \mathrm{d}\pi^2 = 0.
\end{aligned}
$$

We see that the coframing is indeed adapted and that all invariants vanish, except for $T_{3^F} = (-\phi'', 0)$ and $S_{2^F} = (0, -\phi'')$. ⊘

Example 5.2.5. We consider the system $q = r$, $s = G(x, y, u, v)^2 p$. A calculation with MAPLE gives the following adapted coframing:

$$
\begin{aligned}
\theta^1 &= (\mathrm{d}u - p\mathrm{d}x - r\mathrm{d}y) + G^{-1}(\mathrm{d}v - q\mathrm{d}x - (G^2 p)\mathrm{d}y), \\
\theta^2 &= (\mathrm{d}u - p\mathrm{d}x - r\mathrm{d}y) - G^{-1}(\mathrm{d}v - q\mathrm{d}x - (G^2 p)\mathrm{d}y), \\
\omega^1 &= G^{-1}\mathrm{d}x + \mathrm{d}y, \quad \omega^2 = G^{-1}\mathrm{d}x - \mathrm{d}y, \\
\pi^1 &= G\mathrm{d}p + \mathrm{d}r + (pG\tilde{G}_x)\omega^2 + H\theta^2, \quad \pi^2 = G\mathrm{d}p - \mathrm{d}r + 2pG\tilde{G}_x\omega^1 + K\theta^1.
\end{aligned}
$$

Here $\tilde{G}_x = G_x + pG_u + rG_v$, $H = \tilde{G}_y + G\tilde{G}_x + 2pGG_u - 2pG^2G_v$, $K = -\tilde{G}_y + G\tilde{G}_x + 2pGG_u + 2pG^2G_v$. The invariants are $T_{3^F} = 0$, $S_{1^F} = S_{2^F} = 0$, $U_{3^F} = 0$, $V_{2^F} = V_{3^F} = 0$,

$$
\begin{aligned}
T_{2^F} &= (K - pG_u, H - PG_u), \\
U_{2^F} &= \Big(\frac{G_u - GG_v}{4G}, \frac{G_u + GG_v}{4G}\Big)^T.
\end{aligned}
$$

Notice that $S_{2^F} = 0$. In Chapter 7 we will see that S_{2^F} vanishes for all quasi-linear equations. ⊘

Example 5.2.6. Take $q = r$, $s = \phi(p)$ with ϕ an arbitrary function for which $\phi(0) = 0, \phi'(0) = 1$. For convenience we introduce $H = \sqrt{\phi'(p)}$, $K = -(1/8)\phi''(s)/\phi'(s)$. A MAPLE calculation gives the following adapted coframing.

$$
\begin{aligned}
\theta^1 &= (\mathrm{d}u - p\mathrm{d}x - r\mathrm{d}y) + H^{-1}(\mathrm{d}v - q\mathrm{d}x - (G^2 p)\mathrm{d}y), \\
\theta^2 &= (\mathrm{d}u - p\mathrm{d}x - r\mathrm{d}y) - H^{-1}(\mathrm{d}v - q\mathrm{d}x - (G^2 p)\mathrm{d}y), \\
\omega^1 &= \mathrm{d}x + H\mathrm{d}y, \quad \omega^2 = \mathrm{d}x - H\mathrm{d}y, \\
\pi^1 &= \mathrm{d}p + H^{-1}\mathrm{d}r + K\theta^2, \quad \pi^2 = \mathrm{d}p - H^{-1}\mathrm{d}r + K\theta^1.
\end{aligned}
$$

The invariants are $T_{2^F} = 0$, $U_{2^F} = 0$, $V_{2^F} = V_{3^F} = 0$,

$$
\begin{aligned}
T_{3^F} &= (-K, -K)^T, \\
L &= \frac{2\phi'''(p)\phi'(p) - 3\phi''(p)^2}{32\phi'(p)^2}, \\
U_{3^F} &= (L, L)^T, \quad S_{1^F} = (L, L)^T, \quad S_{2^F} = (K, -K)^T.
\end{aligned}
$$

Again S_{1^F}, T_{2^F} and U_{2^F} are zero. The invariant S_{2^F} is unequal to zero, indicating that we are dealing with a non-linear system of equations. ⊘

5.2.2 Contact geometry

If $(M, \mathcal{V})$ is a first order system we can forget about possible base manifolds and only work with the distribution $\mathcal{V}$. The geometry we then have is the geometry invariant under contact transformations. For these systems we can also construct adapted coframings. We start with a slightly bigger structure group, since we do not have to preserve ω modulo θ.

For hyperbolic equations under contact geometry we can find an adapted coframing θ, ω, π such that the contact ideal is given by $I = \operatorname{span}(\theta^1, \theta^2)$ and the structure equations are

$$\begin{aligned} \mathrm{d}\theta = &- \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & \zeta \\ \delta & \epsilon & \alpha - \gamma \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} + \begin{pmatrix} -\pi \wedge \omega \\ 0 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} T_{2F} \omega^F \wedge \theta^F + T_{3F} \pi^F \wedge \theta^F \\ U_{2F} \omega^F \wedge \theta^F + U_{3F} \pi^F \wedge \theta^F \\ V_{2F} \omega^F \wedge \theta^F + V_{3F} \pi^F \wedge \theta^F \end{pmatrix}. \end{aligned} \tag{5.10}$$

We call this an *adapted coframing* for hyperbolic first order systems under contact geometry. The structure group that preserves the form of this adapted coframing is given by

$$x = \begin{pmatrix} a & 0 & 0 \\ b & c & g \\ d & e & f \end{pmatrix}, \tag{5.11}$$

where all coefficients are hyperbolic numbers and $a = cf - eg \in \mathbb{D}^*$. The action of this structure group on the invariants

$$T = (T_{2F}, T_{3F})^T, \quad U = (U_{2F}, U_{3F})^T, \quad V = (V_{2F}, V_{3F})^T, \tag{5.12}$$

is given by the representations

$$\pi_\tau = a^F \begin{pmatrix} c^F & g^F \\ e^F & f^F \end{pmatrix}^T, \quad \begin{aligned} \tilde{T} &= \hat{a} \pi_\tau T, \\ \tilde{U} &= \hat{b} \pi_\tau T + \hat{c} \pi_\tau U + \hat{g} \pi_\tau V, \\ \tilde{V} &= \hat{d} \pi_\tau T + \hat{e} \pi_\tau U + \hat{f} \pi_\tau V. \end{aligned} \tag{5.13}$$

The action on T is important later in the classification of Darboux integrable systems. Note that the action on T is already determined by the action of the group H of matrices

$$\begin{pmatrix} c & g \\ e & f \end{pmatrix} \in \mathrm{GL}(2, \mathbb{D}).$$

5.2.3 Comparison with distributions

In Chapter 4 we have analyzed the structure on the equation manifold of a first order system $(M, \mathcal{V})$ in terms of the distribution $\mathcal{V}$. In the previous sections we have developed a structure theory for the first order systems in terms of adapted coframings on the equation manifold. We will give the correspondence between the different structures for the hyperbolic systems. For elliptic systems similar statements hold.

Let $(M, \mathcal{V})$ be a generalized first order system and let θ, ω, π be an adapted coframing as in (5.10). The Monge systems are given by

$$\operatorname{span}\big(\theta^1, \theta^2, \omega^2, \pi^2\big)^{\perp} \quad \text{and} \quad \operatorname{span}\big(\theta^1, \theta^2, \omega^1, \pi^1\big)^{\perp},$$

respectively. The coefficients $T_{2^F}, T_{3^F}, U_{2^F}, U_{3^F}, V_{2^F}, V_{3^F}$ are the coefficients of the Nijenhuis tensor.

5.2.4 Relations between the invariants

The invariants are not completely independent. We will prove several relations. We start with a lemma about the invariants for systems under contact geometry with a distinguished rank 2 subbundle of $\mathcal{V}$. Every system under point geometry is such a system, since the foliation to the base manifold provides the subbundle.

Definition 5.2.7. A first order system with a *distinguished subbundle* is a first order system $(M, \mathcal{V})$ with a rank 2 subdistribution $\mathcal{W}$ of $\mathcal{V}$. An adapted coframing for such a system is given by an adapted coframing (θ, ω, π) for $(M, \mathcal{V})$ such that the distinguished subdistribution is given by the kernel of $\operatorname{span}(\theta, \omega)$. The structure equations for an adapted coframing are

$$\begin{aligned} \mathrm{d}\theta = &-\begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ \delta & \epsilon & \alpha-\gamma \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \omega \\ \pi \end{pmatrix} + \begin{pmatrix} -\pi\wedge\omega \\ -\pi\wedge\sigma \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} T_{2^F}\omega^F\wedge\theta^F + T_{3^F}\pi^F\wedge\theta^F \\ U_{2^F}\omega^F\wedge\theta^F + U_{3^F}\pi^F\wedge\theta^F \\ V_{2^F}\omega^F\wedge\theta^F + V_{3^F}\pi^F\wedge\theta^F \end{pmatrix}. \end{aligned} \tag{5.14}$$

Here $\sigma = S_{1^F}\theta^F + S_{2^F}\omega^F + S_{3^F}\omega^F$. The structure group is given by

$$x = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & ac^{-1} \end{pmatrix} \in \mathrm{GL}(\mathbb{D}, 3). \qquad \ominus$$

If $S_{3^F} = 0$, then the distinguished subbundle is integrable and provides a foliation. The system is then equivalent to a first order system under point geometry.

Lemma 5.2.8 (First order *T* lemma). *Let θ, ω, π be an adapted coframing* (5.14). *If $T_{3^F} = 0$, then $U_{3^F} + T_{2^F}(S_{3^F})^F = 0$ and $V_{3^F} + T_{2^F}(S_{2^F})^F = 0$.*

Proof. Assume $T_{3^F} = 0$. We calculate $\mathrm{d}^2\theta$ modulo θ, ω, ω^F:

$$\begin{aligned} 0 = \mathrm{d}^2\theta &\equiv \pi\wedge\mathrm{d}\omega + T_{2^F}\mathrm{d}\omega^F\wedge\theta^F \\ &\equiv U_{3^F}\pi\wedge\pi^F\wedge\theta^F - T_{2^F}\pi^F\wedge\sigma^F\wedge\theta^F \\ &\equiv U_{3^F}\pi\wedge\pi^F\wedge\theta^F - T_{2^F}(S_{3^F})^F\pi^F\wedge\pi\wedge\theta^F. \end{aligned}$$

Hence $U_{3^F} + T_{2^F}(S_{3^F})^F = 0$. A similar calculation modulo the differential forms θ, π and ω^F yields $V_{3^F} + T_{2^F}(S_{2^F})^F = 0$. □

Lemma 5.2.9. *Let θ, ω, π be an adapted coframing* (5.14) *with $T_{3^F} = U_{3^F} = V_{3^F} = 0$. Then $T_{2^F}(S_{2^F})^F = 0$ and $T_{2^F}(S_{3^F})^F = 0$. If $T_{2^F} \in \mathbb{D}^*$, then $S_{2^F} = S_{3^F} = 0$ and $S_{1^F} = 0$.*

Proof. By assumption we have $T_{3^F} = U_{3^F} = V_{3^F} = 0$. From the previous lemma it follows that $T_{2^F}(S_{2^F})^F = T_{2^F}(S_{3^F})^F = 0$. If $T_{2^F} \in \mathbb{D}^*$, then by the equations above $S_{2^F} = S_{3^F} = 0$. We calculate $\mathrm{d}^2\omega$ modulo θ, ω, θ^F:

$$\begin{aligned} 0 = \mathrm{d}^2\omega &\equiv -\mathrm{d}\pi \wedge \sigma + \pi \wedge \mathrm{d}\sigma \\ &\equiv \pi \wedge S_{1^F}\mathrm{d}\theta^F \equiv -S_{1^F}\pi \wedge \pi^F \wedge \omega^F. \end{aligned}$$

Hence also $S_{1^F} = 0$. □

Example 5.2.10. Consider the hyperbolic first order system

$$u_y = \frac{(xp+u)^2}{4y}, \quad v_x = \frac{(yb-v)^2}{4x}. \tag{5.15}$$

Let $\psi = -4 + pxsy - pxv + usy - uv$. An adapted coframing is

$$\begin{aligned} \theta^1 &= (4/\psi)\mathrm{d}u - (4p/\psi)\mathrm{d}x - (px+u)^2/\psi\mathrm{d}y, \\ \theta^2 &= (2/\psi)\left(\frac{px+u}{y}\mathrm{d}v - \frac{(px+u)(v-sy)^2}{4}\mathrm{d}x + s(px+u)x\mathrm{d}y\right), \\ \omega^1 &= 4/\psi\mathrm{d}x - 2(px+u)/(y\psi)\mathrm{d}y, \\ \omega^2 &= \frac{-(sy-v)(px+u)}{\psi}\mathrm{d}x + \frac{2(px+u)x}{y\psi}\mathrm{d}y, \\ \pi^1 &= \mathrm{d}p, \quad \pi^2 = \mathrm{d}s. \end{aligned}$$

The almost product structure is integrable. Locally this system is contact equivalent to the first order wave equation. However, for this coframing

$$S_{1^F} = 0, \quad S_{2^F} = \frac{1}{\psi}\begin{pmatrix} 4x/(px+u) \\ y(px+u) \end{pmatrix}.$$

Since S_{2^F} is a relative invariant under point transformations this system cannot be equivalent with the first order wave equation under point transformations. ⊘

5.2.5 Hyperbolic surfaces

A first order system M with projection $\pi : M \to B$ to a base manifold can be embedded in a natural way in $\mathrm{Gr}_2(TB)$. The embedding is given in the Vessiot theorem for first order systems in Section 4.6.1. The fibers of $M \to B$ define surfaces in the fibers of $\mathrm{Gr}_2(TB) \to B$. Invariants of such surfaces are studied in Section 2.3.

Example 5.2.11 (Equation surfaces in the Grassmannian). Let B be a base manifold with coordinates x, y, u, v. The Grassmannian $\mathrm{Gr}_2(TB)$ has local coordinates x, y, u, v, p, q, r, s. We will consider the fiber above the base point $b = (x, y, u, v) \in B$. A point p, q, r, s in the fiber corresponds to the 2-plane in $V = T_b B$ spanned by the vectors $\partial_x + p\partial_u + r\partial_v$ and $\partial_y + q\partial_u + s\partial_v$.

We take $e_1 = \partial_x, e_2 = \partial_y, e_3 = \partial_u, e_4 = \partial_v$ as a basis for $T_b B$. Then the point p, q, r, s corresponds to the element in $\Lambda^2(V)$ given by

$$\begin{aligned}\eta &= (e_1 + pe_3 + re_4) \wedge (e_2 + qe_3 + se_4)\\ &= e_1 \wedge e_2 + qe_1 \wedge e_3 + se_1 \wedge e_4 + pe_3 \wedge e_2 + pse_3 \wedge e_4\\ &\quad + re_4 \wedge e_2 + rqe_4 \wedge e_3.\end{aligned}$$

On $E = \Lambda^2(V)$ we can consider the eigenspaces of the Hodge $*$ operator just as in Section 2.1.2 and introduce Plücker coordinates. The decomposition into the eigenspaces is given by

$$\begin{aligned}\eta = (1 + ps - qr)\alpha_1 + (q + r)\alpha_2 + (s - p)\alpha_3\\ + (1 - ps + qr)\beta_1 + (q - r)\beta_2 + (s + p)\beta_3.\end{aligned} \tag{5.16}$$

Consider the Cauchy-Riemann equations $p = s, q = -r$. The decomposition is given by

$$\eta = (1 + p^2 + r^2)\alpha_1 + (1 - p^2 - r^2)\beta_1 - 2r\beta_2 + 2p\beta_3.$$

Clearly the compactification of the image of the Cauchy-Riemann equation in the oriented Grassmannian $\mathrm{Gr}_2(TM) \cong S^+ \times S^-$ is represented by the graph of the map $S^+ \to S^- : s \mapsto (1, 0, 0)$. The conformal quadratic form defined on the tangent space to the Grassmannian is given by $\xi : \dot\eta \mapsto \dot\eta \wedge \dot\eta$. In the coordinates p, r the conformal quadratic form is

$$\dot\eta \wedge \dot\eta = -4(\dot r^2 + \dot p^2).$$

On the surface defined by the Cauchy-Riemann equation the conformal quadratic form is negative definite and hence the surface is elliptic.

In a similar way we can check that the hyperbolic system $q = r = 0$ has image in $S^+ \times S^-$ given by

$$\eta = (1 + ps)\alpha_1 + (s - p)\alpha^3 + (1 - ps)\beta_1 + (p + s)\beta_3.$$

The conformal quadratic form is $\dot\eta \wedge \dot\eta = -4\dot p\dot s$. ⊘

5.3 Local and microlocal invariants

In Section 2.3 we have analyzed the microlocal invariants of hyperbolic surfaces. We also found some local invariants for hyperbolic first order systems, see sections 5.2.1 and 5.2.2. In this section we will show there is a correspondence between these two kinds of invariants.

For the elliptic case McKay has also derived invariants for an elliptic surface in the Grassmannian. He calls these invariants microlocal invariants and identifies them with the local invariants of the equation manifold. See McKay [51, pp. 28–29, 49–50].

5.3.1 Basic idea

Let $M \subset \mathrm{Gr}_2(TB)$ be a hyperbolic first order system and let us consider the geometry of this system under point transformations. At each point $b_0 \in B$ we have the tangent space $T_{b_0}B$. The fiber M_0 over $b_0 \in B$ is both a submanifold of M and of $\mathrm{Gr}_2(T_{b_0}B)$. As a submanifold of $\mathrm{Gr}_2(T_{b_0}B)$ the surface M_0 defines a hyperbolic surface. The base transformations of the system, i.e., the local diffeomorphisms of B, that fix the point b_0 induce an action on $T_{b_0}B$. This action is given by the tangent map of the transformation and this is a linear transformation. Therefore the action of the point transformations on $M_0 \subset \mathrm{Gr}_2(T_{b_0}B)$ is the action of the general linear group on the hyperbolic surfaces as described in Section 2.3. The microlocal invariants of hyperbolic surfaces must therefore also be invariants of first order systems.

5.3.2 Bundle map

Suppose we are given a first order system as a smooth manifold $M \subset \mathrm{Gr}_2(TB)$ with projection $p : M \to B$. Fix a point $b_0 \in B$ and define M_0 to be $p^{-1}(b_0)$. We write $\iota_M : M_0 \to M$ for the natural embedding of M_0 in M. Recall that M_0 is a hyperbolic surface in $\mathrm{Gr}_2(T_{b_0}B)$ and in Section 2.3.4 we have constructed a bundle B_{M_0} over this surface. In this chapter we have also constructed the bundle $B_M \to M$, which is the bundle of all adapted coframings over M (see Definition 5.2.3). We will construct a bundle map $\iota_M^* B_M \to B_{M_0}$ over M_0 that will identify the local invariants with the microlocal invariants.

A point f in B_M is a point $m \in M$ together with an adapted coframing θ, ω, π for $T_m M$. Fix a point $m_0 \in M_0 = p^{-1}(b_0)$ and consider the derivative $T_{m_0}p : T_{m_0}M \to T_{b_0}B$. The kernel of $T_{m_0}p$ is given by the vectors at m_0 that satisfy $\theta = \omega = 0$. This implies that the 1-forms θ, ω are mapped to 1-forms $\Theta(m_0)$, $\Omega(m_0)$ on $T_{b_0}M$. The forms Θ, Ω depend on the point m_0 chosen. Note that the value of $\Theta(m_0)$ and $\Omega(m_0)$ does only depend on the value of the coframe θ, ω, π at m_0 and not on the derivatives of the coframe. Therefore we can also write $\Theta(f_0)$, $\Omega(f_0)$ for the linear maps $T_{b_0}B \to \mathbb{D}$ that depend on a point f_0 in the bundle of adapted coframes B_M over M.

The pair (Ω, Θ) gives a linear map $T_{b_0}B \to \mathbb{R}^4$. If we let $V = T_{b_0}B$ and identify $\mathbb{R}^4$ with V, then this defines an element of $\mathrm{GL}(V)$. This construction gives us a map μ from $\iota_M^*(B_M)$ to B_0 which is a bundle map over M_0. Next we want to prove that μ maps not only into B_0 but in fact into $B_{M_0} \subset B_0$ and that the microlocal invariants (functions on B_{M_0}) correspond to local invariants (functions on B_M).

The bundle map is is written in the diagram below. We let $V = T_{b_0}B$.

$$\begin{tikzcd}
B_{M_0} \arrow[r, hook] & B_0 \arrow[r] \arrow[d] & B_V \arrow[d] \\
\iota_M^* B_M \arrow[u, dotted, "\mu?"] \arrow[ur, "\mu"] \arrow[r] \arrow[d] & M_0 \arrow[r, hook, "\iota"] \arrow[d, hook, "\iota_M"] & \mathrm{Gr}_2(V) \\
B_M \arrow[r] & M &
\end{tikzcd}$$

The dimension of B_M is 16, the base manifold M is 6-dimensional and we have a 10-dimensional structure group. The pullback bundle $\iota_M^*(B_M)$ has dimension 12 (four less than B_M). The map μ maps this bundle to B_{M_0} which has dimension 8. Hence the map has a 4-dimensional kernel. Four degrees of freedom are lost since the structure group of $\iota_M^*(B_M)$ contains transformations of the coframe where π is changed and the bundle map μ does not depend on π.

5.3.3 The local-microlocal dictionary

We claim that the map μ is in fact a bundle map $\iota_M^* B_M \to B_{M_0}$. A proof will be given in the next section.

Let $V = T_{b_0}B$. On the pullback bundle $\iota_M^* B_M$ we have at each point f an adapted coframing (θ, ω, π) for the tangent space to M_0 at the point m (m is the base point of f for the bundle $\iota_M^* B_M \to M_0$). Under the map μ the components $\xi, \eta, \vartheta, \zeta$ of the Maurer-Cartan form on B_0 are pulled back to

$$\begin{aligned}
\xi' &= -\gamma, & \xi'' &= -S_{2^F}\pi, \\
\eta' &= -\beta, & \eta'' &= U_{3^F}\pi^F - S_{1^F}\pi, \\
\vartheta' &= -\pi, & \vartheta'' &= 0, \\
\zeta' &= -\alpha, & \zeta'' &= T_{3^F}\pi^F.
\end{aligned} \tag{5.17}$$

In particular the microlocal invariants f and g correspond to the local invariants S_{2^F} and T_{3^F}.

5.3.4 Proof of the correspondence

Note that we have defined the map μ as a map from $B_M \to B_0$, but in fact μ is already defined as a map from the bundle of unadapted coframings θ, ω, π to B_0. We will prove that the adaptations from an unadapted coframing on M to an adapted coframing correspond to the normalizations of the Maurer-Cartan form in the microlocal setting.

We start with the bundle $\tilde{B}_M$ of coframes for which the kernel of span(θ) is equal to the contact distribution and the kernel of span(θ, ω) is tangent to the projection $M \to B$. The sections of this bundle are the unadapted coframes for the first order systems. Let (θ, ω, π) be the pullback of the soldering form to $\tilde{B}_M$. We start with the structure equations for the components θ and ω.

$$\begin{aligned}
\mathrm{d}\theta &= -\alpha \wedge \theta - c_{1,1}\pi \wedge \omega - c_{1,F}\pi \wedge \omega^F - c_{F,1}\pi^F \wedge \omega - c_{F,F}\pi^F \wedge \omega^F + \tau_1 \wedge \theta^F, \\
\mathrm{d}\omega &= -\beta \wedge \theta - \gamma \wedge \omega - \pi \wedge \sigma + d_{F,F}\pi^F \wedge \omega^F + \tau_2 \wedge \theta^F.
\end{aligned}$$

Here $\sigma = S_{1^F}\theta^F + S_{2^F}\omega^F$, $\tau_1 = T_2\omega + T_{2^F}\omega^F + T_3\pi + T_{3^F}\pi^F$ and $\tau_2 = U_2\omega + U_{2^F}\omega^F + U_3\pi + U_{3^F}\pi^F$. These are the most general structure equations for which $\mathrm{d}\theta \equiv \mathrm{d}\omega \equiv 0 \mod \theta, \theta^F, \omega, \omega^F$.

The map μ is a map from $\iota_M^* \tilde{B}_M$ to B_0. We calculate how μ varies when we move along a fiber of $\iota_M^* \tilde{B}_M \to B$. Let Z be a vector field on $\iota_M^* \tilde{B}_M$ tangent to the fibers of $\iota_M^* \tilde{B}_M \to B$.

Let X be a vector field on B and take a lift $\tilde{X}$ to $\iota_M^* \tilde{B}_M$ such that $\pi(\tilde{X}) = 0$. Let f_t be equal to $\exp(tZ) f_0$. Then we have

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Theta(f_t)(X) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta_{f_t}(\tilde{X}) = \mathcal{L}_Z(\theta(\tilde{X})) = (\mathcal{L}_Z\theta)(\tilde{X}) + \theta(\mathcal{L}_Z\tilde{X}) \\
&= (\mathcal{L}_Z\theta)(\tilde{X}) = \mathrm{d}\theta(Z, \tilde{X}) \\
&= -\alpha(Z)\theta(\tilde{X}) - \sum_{i,j=1,F} c_{i,j}\pi^i(Z)\omega^j(\tilde{X}) + \tau_1(Z)\theta^F(\tilde{X}) \\
&= -\alpha(Z)\Theta(f_0)(X) - \sum_{i,j=1,F} c_{i,j}\pi^i(Z)\Omega^j(f_0)(X) \qquad (5.18) \\
&\qquad + \tau_1(Z)\Theta(f_0)^F(X), \\
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Omega(f_t)(X) &= -\beta(Z)\Theta(f_0)(X) - \gamma(Z)\Omega(f_0)(X) \\
&\qquad - \pi(Z)\sigma(\tilde{X}) + d_{F,F}\pi^F(Z) \wedge \omega^F(\tilde{X}) + \tau_2(Z)\theta^F(\tilde{X}).
\end{aligned}$$

The map μ can be viewed as a map $\iota_M^* \tilde{B}_M \to \mathrm{GL}(V)$. Then $\mathrm{d}\mu$ is a 1-form on $\iota_M^* B_M$ that takes values in $\mathrm{Lin}(V, V)$. We can then define $(\mathrm{d}\mu)\mu^{-1}$ as the multiplication of $\mathrm{d}\mu$ with $(\mu)^{-1}$. This is a 1-form on the pullback $\iota_M^* \tilde{B}_M$ of the bundle of coframes on M, valued in the linear maps from V to V. From the structure equations (5.18) we can read off that

$$(\mathrm{d}\mu)\mu^{-1} = \begin{pmatrix} -\gamma - S_2\pi + d_{F,F}\pi^F L & -\beta - S_1\pi L + U_{3^F}\pi^F L \\ -c_{1,1}\pi - c_{F,1}\pi^F - c_{1,F}\pi L - c_{F,F}\pi^F L & -\alpha + T_{3^F}\pi^F L + T_{3^F}\pi L \end{pmatrix}. \tag{5.19}$$

On the other hand the 1-form $(\mathrm{d}\mu)\mu^{-1}$ is equal to the pullback of the Maurer-Cartan form on B_0 to $\iota_M^* \tilde{B}_M$. So the components of $(\mathrm{d}\mu)\mu^{-1}$ correspond to the pullbacks of the components of the Maurer-Cartan form.

The first normalization on the microlocal level is arranging $\vartheta'' = 0$. We can see that this corresponds to making the terms $c_{1,F}$ and $c_{F,F}$ zero. The first step in the adaptation of the local coframe is to make $\mathrm{d}\theta \equiv -\pi \wedge \omega \mod \theta, \theta^F$. This corresponds to making $c_{F,1}$, $c_{1,F}$ and $c_{F,F}$ zero. So the first step in the adaptation of the local coframing implies the first normalization at the microlocal level. This proves that the map $\mu : \iota_M^* B_M \to B_0$ descends to a map $\mu : \iota_M^* B_M \to B_1$.

We can continue and check that for each normalization at the microlocal level, there is a corresponding normalization at the local level. Note that at the local level there are more steps, for example the term $c_{F,1}$ is eliminated at the local level, but is not present at the microlocal level. At the end we have the bundle of adapted coframings at the local level, the bundle B_{M_0} at the microlocal level and a map $\mu : \iota_M^* B_M \to B_{M_0}$.

5.4 Elliptic systems

In this section we discuss the structure theory for elliptic first order systems. Since the structure theory for hyperbolic systems was already discussed in detail in Section 5.2.1, we only give the final results for the adapted coframe.

5.4.1 Structure equations

Recall that (see Section 3.3) any elliptic first order system with a projection to a base manifold B can be described as a 6-dimensional manifold M with contact forms θ^1, θ^2 and forms ω^1, ω^2 describing the projection to B.

We can complete these 1-forms to a complex coframe $\Phi = (\theta, \omega, \pi)^T$ on M. In McKay [51] it is shown that the coframe can be chosen such that the following structure equations are satisfied

$$\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = -\begin{pmatrix}\alpha & 0 & 0\\ \beta & \gamma & 0\\ \delta & \epsilon & \alpha-\gamma\end{pmatrix}\wedge\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} - \pi\wedge\begin{pmatrix}\omega\\ \sigma\\ 0\end{pmatrix} + \begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\end{pmatrix}\wedge\bar{\theta}, \tag{5.20}$$

where

$$\begin{aligned}\sigma &= S_{\bar{1}}\bar{\theta} + S_{\bar{2}}\bar{\omega},\\ \tau_1 &= T_{\bar{2}}\bar{\omega} + T_{\bar{3}}\bar{\pi}, \quad \tau_2 = U_{\bar{2}}\bar{\omega} + U_{\bar{3}}\bar{\pi}, \quad \tau_3 = V_{\bar{2}}\bar{\omega} + V_{\bar{3}}\bar{\pi}.\end{aligned} \tag{5.21}$$

Let G be the structure group preserving this adapted coframing, i.e., G is the group of lower triangular matrices with diagonal entries (a, c, ac^{-1}). Let

$$\begin{aligned} g &= \begin{pmatrix}a & 0 & 0\\ b & c & 0\\ d & e & ac^{-1}\end{pmatrix},\\ g^{-1} &= \begin{pmatrix}a^{-1} & 0 & 0\\ -ba^{-1}c^{-1} & c^{-1} & 0\\ a^{-2}be - a^{-2}cd & -a^{-1}e & a^{-1}c\end{pmatrix} = \begin{pmatrix}\hat{a} & 0 & 0\\ \hat{b} & \hat{c} & 0\\ \hat{d} & \hat{e} & \hat{a}\hat{c}^{-1}\end{pmatrix}.\end{aligned} \tag{5.22}$$

We can calculate the action the structure group G on the invariants just as in Section 5.2.1. Define the following representations of G: $\pi_\tau : G \to \mathrm{Aut}(\mathbb{C}^2)$, $\pi_\sigma : G \to \mathrm{Aut}(\mathbb{C}^2)$ by

$$\pi_\tau(g) = \bar{a}\begin{pmatrix}\bar{c} & \bar{e}\\ 0 & \bar{a}\bar{c}^{-1}\end{pmatrix}, \tag{5.23}$$

$$\pi_\sigma(g) = ac^{-1}\begin{pmatrix}\bar{a} & \bar{b}\\ 0 & \bar{c}\end{pmatrix}. \tag{5.24}$$

Under the action of G the invariants $S = (S_{\bar{1}}, S_{\bar{2}})$ transform under the representation $\hat{c}\pi_\sigma$ and the invariants T, U, V transform as

$$\begin{aligned}\tilde{T} &= a^{-1}\pi_\tau T,\\ \tilde{U} &= \hat{b}\pi_\tau T + \hat{c}\pi_\tau U,\\ \tilde{V} &= \hat{d}\pi_\tau T + \hat{e}\pi_\tau U + a^{-1}c\pi_\tau V.\end{aligned}$$

5.4.2 Transformations of the base foliation

On the equation manifold M the contact forms θ^1, θ^2 together with the independence forms ω^1, ω^2 determine an elliptic system. The system is determined by the two bundles $I = \operatorname{span}(\theta^j)$ and $J = \operatorname{span}(\theta^j, \omega^k)$. Any (infinitesimal) group of internal symmetries must map I to a bundle $\tilde{I} = I$ and J to a new bundle $\tilde{J}$. If $J = \tilde{J}$, then we are dealing with a prolonged point transformation and hence an external symmetry. The internal contact symmetries that cannot be extended to external contact symmetries are characterized by transformations for which $J \neq \tilde{J}$.

Let us analyze the transformations of the foliation defined by J. For both the old pair (I, J) and the new pair $(I, \tilde{J})$ we can choose adapted coframing θ, ω, π and $\tilde{\theta}, \tilde{\omega}, \tilde{\pi}$. When we express the structure equations for this new coframing using the structure equations of the old coframing, we can determine conditions under which the new coframing defines the same system.

First note that since $\tilde{I} = I$ we know that $\tilde{\theta} = c\theta$, $c \in \mathbb{C}^*$ (this follows from the fact that θ in an adapted coframing is completely determined up to a positive scalar factor). Using the structure group $\tilde{G}$ we can arrange $c = 1$, i.e., $\tilde{\theta} = \theta$.

From the contact structure group (the hyperbolic version is given in equation (5.11)), it follows that our transformation is of the form

$$\begin{aligned}\tilde{\theta} &= \theta,\\ \tilde{\omega} &= A_1\theta + A_2\omega + A_3\pi,\\ \tilde{\pi} &= B_1\theta + B_2\omega + B_3\pi,\end{aligned}$$

with $A_2B_3 - A_3B_2 = 1$. Using two more degrees of freedom in the structure group we can arrange $A_1 = B_1 = 0$. So for any transformation mapping the equation to a new first order equation we can arrange that the transformation of the adapted coframe is of the form

$$\begin{aligned}\tilde{\theta} &= \theta,\\ \tilde{\omega} &= A_2\omega + A_3\pi,\\ \tilde{\pi} &= B_2\omega + B_3\pi.\end{aligned} \tag{5.25}$$

We analyze how the transformation defined above acts on the differential invariants. We calculate

$$
\begin{aligned}
\mathrm{d}\tilde{\theta} = \mathrm{d}\theta &= -\alpha \wedge \theta - \pi \wedge \omega + \tau_1 \wedge \bar{\theta} \\
&= -\alpha \wedge \tilde{\theta} - (-B_2\tilde{\omega} + A_2\tilde{\pi}) \wedge (B_3\tilde{\omega} - A_3\tilde{\pi}) \\
&\quad + T_{\bar{2}}(\overline{B_3}\bar{\tilde{\omega}} - \bar{A}_3\bar{\tilde{\pi}}) \wedge \bar{\tilde{\theta}} + T_{\bar{3}}(-\overline{B_2}\bar{\tilde{\omega}} + \overline{A_2}\bar{\tilde{\pi}}) \wedge \bar{\tilde{\theta}} \\
&= -\tilde{\alpha} \wedge \tilde{\theta} - \tilde{\pi} \wedge \tilde{\omega} + \tilde{\tau}_1 \wedge \bar{\tilde{\theta}},
\end{aligned}
$$

with

$$
\begin{gathered}
\tilde{\alpha} = \alpha, \quad \tilde{\tau}_1 = \widetilde{T_{\bar{2}}}\bar{\tilde{\omega}} + \widetilde{T_{\bar{3}}}\bar{\tilde{\pi}}, \\
\widetilde{T_{\bar{2}}} = \overline{B_3}T_{\bar{2}} - \overline{B_2}T_{\bar{3}}, \quad \widetilde{T_{\bar{3}}} = \overline{A_2}T_{\bar{3}} - \overline{A_3}T_{\bar{2}}.
\end{gathered}
$$

We define the representation

$$
\begin{gathered}
\rho_\tau : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Aut}(\mathbb{C}^2) : \\
h = \begin{pmatrix} A_2 & A_3 \\ B_2 & B_3 \end{pmatrix} \mapsto \begin{pmatrix} \overline{B_3} & -\overline{B_2} \\ -\overline{A_3} & \overline{A_2} \end{pmatrix}.
\end{gathered}
$$

The transformation of the base foliation will act on the invariants T as $\tilde{T} = \rho_\tau(h)T$. In a similar way we can derive the action of the transformation on the other invariants as

$$
\begin{aligned}
\tilde{T} &= \rho_\tau(h)T, \\
\tilde{U} &= A_2\rho_\tau(h)U + A_3\rho_\tau(h)V, \\
\tilde{V} &= B_2\rho_\tau(h)U + B_3\rho_\tau(h)V.
\end{aligned} \tag{5.26}
$$

The invariants S do not transform properly (the transformed variables $\tilde{S}$ do not only depend on the values of A_2, A_3, B_2 and B_3, but on the first order derivatives of A_2, A_3, B_2, B_3 as well).

Notice that in order for the transformation to define a new foliation by solutions we should have $\tilde{\sigma} \equiv 0 \mod \bar{\tilde{\theta}}, \bar{\tilde{\omega}}$. This gives restrictions on the functions A_j, B_j that are allowed. In principle these can be expressed in terms of $\theta, \omega, \pi, \alpha, \gamma, \epsilon$. In coordinate invariant terms these restrictions are $\mathrm{d}\tilde{J} \equiv 0 \mod \tilde{J}$. The new system $(I, \tilde{J})$ gives a projection to a new base manifold $\tilde{B}$. This defines a new first order system and we can ask the question if this new system is equivalent to the old system by a point transformation.

Theorem 5.4.1. *Let M be an elliptic first order system with base manifold with invariants $T_{\bar{2}} \neq 0$, $T_{\bar{3}} = 0$. Then all internal symmetries are induced from base transformations and hence are point symmetries (and hence external symmetries).*

Proof. If we have a transformation that acts non-trivial on the foliation we must have $A_3 \neq 0$. Notice that $\widetilde{T_{\bar{3}}} = \overline{A_2}T_{\bar{3}} - \overline{A_3}T_{\bar{2}} = -\overline{A_3}T_{\bar{2}}$. Since $\overline{A_3} \neq 0$ and $\widetilde{T_{\bar{3}}} \neq 0$ this implies $\widetilde{T_{\bar{3}}} \neq 0$. But $T_{\bar{3}}$ is a relative invariant under point geometry and hence the new equation cannot be equivalent to the original one. □

5.4.3 Examples

Example 5.4.2 (A class of equations with only point symmetries). Consider the generalized elliptic systems defined by $s = p$, $q = -r + F(x, y, u, v)$. From the contact forms $\theta^1 = \mathrm{d}u - p\mathrm{d}x + (r - F)\mathrm{d}y$, $\theta^2 = \mathrm{d}v - r\mathrm{d}x - p\mathrm{d}y$ and the independence forms $\omega^1 = \mathrm{d}x$, $\omega^2 = \mathrm{d}y$ we can choose the following complex coframing

$$\begin{aligned}
\theta &= \theta^1 + i\theta^2, \\
\omega &= \omega^1 + i\omega^2 = \mathrm{d}x + i\mathrm{d}y, \\
\pi &= \mathrm{d}p + i(\mathrm{d}r - \tilde{F}_x\mathrm{d}x) - (i/4)(F_u + iF_v)\bar{\theta},
\end{aligned}$$

where $\tilde{F}_x = F_x + pF_u$. By adding the term $-\tilde{F}_x\mathrm{d}x$ to π we have the structure equations $\mathrm{d}\theta \equiv -\pi \wedge \omega \mod \theta, \bar{\theta}$ and the term $-(i/4)(F_u + iF_v)\bar{\theta}$ completes the adaptation.

The structure equations for θ, ω and π are

$$\begin{aligned}
\mathrm{d}\theta &= -(-(1/2)(F_u - iF_v)\omega^2) \wedge \theta - \pi \wedge \omega + (i/4)(F_u + iF_v)\bar{\omega} \wedge \bar{\theta}, \\
\mathrm{d}\omega &= 0, \\
\mathrm{d}\pi &= -\delta \wedge \theta - \epsilon \wedge \omega - (\alpha - \gamma) \wedge \pi + \tau_3 \wedge \bar{\theta},
\end{aligned}$$

for certain 1-forms δ, ϵ and $\tau_3 = V_{\bar{2}}\bar{\omega}$. We arrive at $T_{\bar{2}} = (i/4)(F_u + iF_v)$, $T_{\bar{3}} = 0$. From Theorem 5.4.1 we conclude if if either $F_u \neq 0$ or $F_v \neq 0$ then every internal symmetry is the prolongation of a base transformation. $\oslash$

Example 5.4.3 (Cauchy-Riemann equations). Consider the Cauchy-Riemann equations $s = p, q = -r$. An adapted coframing is given by $\theta = (\mathrm{d}u - p\mathrm{d}x + r\mathrm{d}y) + i(\mathrm{d}v - r\mathrm{d}x - p\mathrm{d}y)$, $\omega = \mathrm{d}x + i\mathrm{d}y$, $\pi = \mathrm{d}p + i\mathrm{d}r$. The exact structure equations are

$$\mathrm{d}\theta = -\pi \wedge \omega, \quad \mathrm{d}\omega = 0, \quad \mathrm{d}\pi = 0.$$

We will consider transformations of the coframing as described previously. We take the element

$$\begin{pmatrix} A_2 & A_3 \\ B_2 & B_3 \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

The new coframe $\tilde{\theta} = \theta$, $\tilde{\omega} = \cos(\phi)\omega - \sin(\phi)\pi$, $\tilde{\pi} = \sin(\phi)\omega + \cos(\phi)\pi$ has the same structure equations. This transformation therefore provides a good candidate for a 1-parameter family of internal symmetries of the Cauchy-Riemann equations. The new forms $\tilde{\theta}$, $\tilde{\omega}$ provide a new foliation of the manifold M. The tangent spaces to the leaves of this foliation are spanned by

$$\begin{aligned}
\partial_{\tilde{\pi}_1} &= \cos(\phi)\partial_p + \sin(\phi)(\partial_x + p\partial_u + r\partial_v), \\
\partial_{\tilde{\pi}_2} &= \cos(\phi)\partial_r + \sin(\phi)(\partial_y - r\partial_u + p\partial_v).
\end{aligned}$$

For the quotient manifold of the first order system and this foliation we can use the coordinates $\tilde{x} = \cos(\phi)x - \sin(\phi)p$, $\tilde{y} = \cos(\phi)y - \sin(\phi)r$,

$$\tilde{u} = u + (1/2)\tan(\phi)r^2 - (1/2)\tan(\phi)p^2,$$
$$\tilde{v} = v - \tan(\phi)pr.$$

Using the projection map $\pi : M \mapsto \tilde{B} : (x, y, u, v, p, r) \mapsto (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ we can construct a new first order system. The distribution $\mathcal{V}$ dual to θ is mapped under $T\pi$ to the 2-dimensional bundle

$$\operatorname{span}\left(\cos(\phi)\partial_{\tilde{x}} + p\partial_{\tilde{u}} + r\partial_{\tilde{v}}, \cos(\phi)\partial_{\tilde{y}} - r\partial_{\tilde{u}} + p\partial_{\tilde{v}}\right).$$

This map defines a contact transformation $M \to \tilde{M} \subset \mathrm{Gr}_2(T\tilde{B})$ by $(x, y, u, v, p, r) \mapsto (\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{p} = p/\cos(\phi), \tilde{q} = -r/\cos(\phi), \tilde{r} = r/\cos(\phi), \tilde{s} = p/\cos(\phi))$. The image is clearly the 2-dimensional surface in the Grassmann bundle of $\tilde{M}$ that corresponds to the Cauchy-Riemann equations on $\tilde{M}$. ⊘

5.5 Miscellaneous

5.5.1 Hyperbolic exterior differential systems

Both Bryant et al. [14, 15] and Vassiliou [66] have defined the concept of a hyperbolic exterior differential system.

Definition 5.5.1. A *hyperbolic exterior differential system* of class s is an exterior differential system $(M, \mathcal{I})$ on a manifold M of dimension $s + 4$ such that there exists a coframing $\theta^1, \ldots, \theta^s, \omega^1, \ldots, \omega^4$ such that

$$\mathcal{I} = \left\{\theta^1, \ldots, \theta^s, \omega^1 \wedge \omega^3, \omega^2 \wedge \omega^4\right\}_{\mathrm{alg}}.$$ ⊖

Such a coframing is called an admissible local coframing. A more intrinsic definition is given in Bryant et al. [14, p. 31] but the definition above is enough for our purposes. Vassiliou defines (Definition 2.10 in Vassiliou [66]) a manifold of (p, q)-hyperbolic type as a hyperbolic exterior differential system with some additional conditions. These additional conditions allow him to treat the manifolds of (p, q)-hyperbolic type as generalizations of Darboux integrable equations with p or q invariants for each characteristic system. We will give examples of hyperbolic exterior differential systems with class 0, 1 and 2. These examples will make clear that the theory of hyperbolic exterior differential systems is very relevant to our first order systems and second order equations.

Let us give an alternative definition of the hyperbolic exterior differential systems. Let $(M, \mathcal{I})$ be a hyperbolic exterior differential system as in Definition 5.5.1. The distribution $\mathcal{W}$ dual to the 1-forms $\theta^1, \ldots, \theta^s$ is a rank 4 distribution on M. The 2-forms $\omega^1 \wedge \omega^3$, $\omega^2 \wedge \omega^4$ restrict to two non-zero two forms on $\mathcal{W}$. We define $\mathcal{W}_+ = \ker(\omega^2 \wedge \omega^4|_{\mathcal{W}})$ and $\mathcal{W}_- = \ker(\omega^1 \wedge \omega^3|_{\mathcal{W}})$. The two rank two distributions $\mathcal{W}_\pm$ define a hyperbolic structure on $\mathcal{W}$ (up to a change of the characteristic systems). Conversely, a rank 4 distribution $\mathcal{W}$ on a $(4+s)$-dimensional manifold with a hyperbolic structure on $\mathcal{W}$ defines a hyperbolic manifold of class s. The 2-dimensional integral elements for $(M, \mathcal{W}_+, \mathcal{W}_-)$ at a point x are the 2-dimensional linear subspaces E of $\mathcal{W}_x$ for which $E \cap (\mathcal{W}_+)_x$ and $E \cap (W_-)_x$ are 1-dimensional. In particular $(\mathcal{W}_+)_x$ and $(\mathcal{W}_-)_x$ are not integral elements of the system. We can also say that the 2-dimensional integral elements at x are the hyperbolic lines for the hyperbolic structure on $\mathcal{W}_x$. For this reason we can also give the following alternative definition of an hyperbolic exterior differential system.

Definition 5.5.2 (Alternative definition of hyperbolic exterior differential system). A hyperbolic exterior differential system of class s is a manifold M of dimension $4+s$ together with a rank 4 distribution $\mathcal{W}$ with a hyperbolic structure K on $\mathcal{W}$. We write the system as $(M, \mathcal{W}_+, \mathcal{W}_-)$. ⊖

Example 5.5.3. On $\mathbb{R}^4$ with coordinates x, y, p, q define the hyperbolic exterior differential system

$$\mathcal{I} = \{\mathrm{d}x \wedge \mathrm{d}p, \mathrm{d}y \wedge \mathrm{d}q\}_{\text{diff}}. \tag{5.27}$$

In terms of the alternative definition Definition 5.5.2 we can give the hyperbolic exterior differential system by the two characteristic systems $\mathcal{W}_+ = \mathrm{span}(\partial_x, \partial_p)$, $\mathcal{W}_- = \mathrm{span}(\partial_y, \partial_q)$. These two distributions define an integrable almost product structure on M. The integral manifolds of $\mathcal{I}$ are precisely the hyperbolic pseudoholomorphic curves for this integrable almost product structure. ⊘

Example 5.5.4 (Monge-Ampère equations). Consider the Monge-Ampère equation

$$E(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D, \tag{5.28}$$

where A, B, C, D and E are functions of the first order variables $x, y, z, p = z_x, q = z_y$. The graphs of 1-jets of solutions of the equation correspond to integral surfaces of the exterior differential system $\mathcal{I}$ on $\mathrm{J}^1(\mathbb{R}^2, \mathbb{R})$ generated by

$$\begin{aligned}
\theta &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\
\Psi &= E\mathrm{d}p \wedge \mathrm{d}q + A\mathrm{d}p \wedge \mathrm{d}q + B(\mathrm{d}q \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}p) \\
&\quad + C\mathrm{d}x \wedge \mathrm{d}q + D\mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

If the Monge-Ampère equation is hyperbolic, then $\mathcal{I}$ defines a hyperbolic exterior differential system of class $s = 1$. ⊘

Example 5.5.5 (Hyperbolic exterior differential systems of class $s = 2$). Every hyperbolic first order system $(M, \mathcal{V})$ defines a hyperbolic exterior differential system of class $s = 2$. If θ, ω, π is an adapted coframing, then the exterior differential ideal $\mathcal{I}$ is generated by $\{\theta^1, \theta^2, \mathrm{d}\theta^1, \mathrm{d}\theta^2\}_{\text{alg}}$. Note that the contact distribution $\mathcal{V}$ of the first order system corresponds to the distribution $\mathcal{W}$ determined by $\mathcal{I}$.

The converse is not true. A general hyperbolic exterior differential system of class $s = 2$ might have an integrable distribution $\mathcal{W}$. The conditions for the distribution $\mathcal{W}$ to determine a first order system are described in Theorem 4.6.4. ⊘

Example 5.5.6 (Second order scalar equations). Also the Vessiot systems $(M, \mathcal{V})$ associated to hyperbolic second order scalar equations are hyperbolic exterior differential systems, but this time of class $s = 3$. The conditions on $\mathcal{V}$ imply that not every hyperbolic exterior differential system of class $s = 3$ corresponds to a second order partial differential equation. $\oslash$

The prolongation of a hyperbolic exterior differential system of class k is a hyperbolic exterior differential system of class $k + 2$. This proved for example in Bryant and Griffiths [16, Section 1.3]. This means that if we are studying prolongations of first order systems or second order scalar equations, we are still studying hyperbolic exterior differential systems.

As an elliptic analogue we can define

Definition 5.5.7 (Elliptic exterior differential system). Let M be a manifold of dimension $4 + s$. Let $\mathcal{V}$ be a rank 4 distribution on M and $J : \mathcal{V} \to \mathcal{V}$ a complex structure. The pair $(M, \mathcal{V}, J)$ is called an *elliptic exterior differential system* of class s. $\ominus$

The equations for pseudoholomorphic curves in a 4-dimensional almost complex manifold, the elliptic first order systems and elliptic second order scalar partial differential equations are examples of elliptic exterior differential systems of class $s = 0$, $s = 2$ and $s = 3$, respectively.

5.5.2 Linear hyperbolic systems

Let E, F be two rank 2 vector bundles over a smooth 2-dimensional manifold X. Consider a hyperbolic linear partial differential operator $K : C^\infty(E) \to C^\infty(F)$. In local coordinates $x = (x^1, x^2)^T$ for X, $u = (u^1, u^2)^T$ for E and $f = (f^1, f^2)^T$ for F the operator K is given by

$$Ku = A^1 \frac{\partial u}{\partial x^1} + A^2 \frac{\partial u}{\partial x^2} + Bu.$$

Here A^1, A^2, B are 2×2-matrices. The type of the operator K is determined by the principal symbol of K, which is defined as the matrix $\sigma(x, \xi) = A^1(x)\xi_1 + A^2(x)\xi_2$. The operator K is hyperbolic or elliptic if $\det \sigma(x, \xi)$ is a quadratic form in ξ with positive or negative discriminant, respectively. Transformations of the coordinates of the form $x = x(y)$, $u = S(y)v$, $f = T(y)g$ act on the principal symbol as

$$(\tilde{A}^\beta)^p_m = (T^{-1})^p_k (A^\alpha)^k_l S^l_m \frac{\partial y^\beta}{\partial x^\alpha}.$$

Under this action the sign of the discriminant of the quadratic form is unchanged and hence the type of operator is unchanged. Using this action (the action of T and S is in fact the action of the conformal group CO(2, 2), see Appendix A.5) we can arrange that

$$A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Next we will transform the operator K into a normal form. The normal form is inspired by a corresponding normal form in the elliptic case, see Duistermaat [23]. The complex variables are replaced by hyperbolic variables, i.e., elements of the algebra $\mathbb{D}$ introduced before. After transformation of the principal symbol into normal form the operator is given by

$$Ku = \frac{\partial u}{\partial x^F} + a(x, x^F)u + b(x, x^F)u^F.$$

Here we have written symbolically $\partial u/\partial x^F = A^1 \partial u^2/\partial x^1 + A^2 \partial u^1/\partial x^2$ and a, b are $\mathbb{D}$-valued functions of the base variables x, x^F. Next we solve the system of ordinary differential equations for the $\mathbb{D}$-valued function c

$$\frac{\partial c}{\partial x^F} = a(x, x^F), \quad c(0) = 0.$$

We apply the transformation $u(x) = \exp(-c(x, x^F))\tilde{u}(x)$ (the exponential of a $\mathbb{D}$-valued function is defined as a formal power series, or equivalently as the exponential applied to the components). In the new coordinates we have

$$\tilde{K}\tilde{u} = Ku = \exp(-c(x))\frac{\partial \tilde{u}}{\partial x^F} + b(x, x^F)\exp(-c(x))\tilde{u}^F.$$

By applying a scale transformation with $\exp(-c)$ we arrive at the normal form (we drop the tilde in the notation)

$$Ku = \frac{\partial u}{\partial x^F} + b(x, x^F)u^F. \tag{5.29}$$

Remark 5.5.8. The principal symbol of K is a linear map

$$\sigma(x, \xi) : E_x \to F_x$$

that is linear in $\xi \in (T_x X)^*$. The condition that K is hyperbolic is precisely that the conformal quadratic form $\xi \mapsto \det \sigma(x, \xi)$ is non-degenerate indefinite. The equation $\det \sigma(x, \xi) = 0$ then has two real characteristic solutions $\zeta_{1,2} \in (T_x X)^*$. The dual distributions to $\zeta_{1,2}$ determine an almost product structure on X. Since $\sigma(x, \zeta_j)$ is of rank 1 the distributions $\mathcal{E}_j = \ker \sigma(x, \zeta_j)$ and $\mathcal{F}_j = \operatorname{im} \sigma(x, \zeta_j)$ determine almost product structures on the bundles $E \to X$ and $F \to X$. ⊘

Example 5.5.9. We use the normal form of a linear equation to analyze the invariants. For convenience we take the normal form $\partial w/\partial z^F + b(z, z^F)w^F = 0$ which is better adapted to our previous notations, but is equivalent to the normal form (5.29) by a simple substitution of the variables $w \mapsto u$, $z \mapsto x$. The equation written down explicitly in the coordinates $w = (u, v)^T$, $z = (x, y)^T$ is

$$u_y + b^1 v = 0, \quad v_x + b^2 u = 0.$$

We define $q = -bw^F$, $p = \partial_z w = (u_x, v_y)^T$ and the coframing

$$\begin{aligned} \theta &= \mathrm{d}w - p\mathrm{d}z - q\mathrm{d}z^F, \\ \omega &= \mathrm{d}z, \\ \pi &= \mathrm{d}p + (b_z w^F + bq^F)\omega^F. \end{aligned} \tag{5.30}$$

The structure equations are

$$\begin{aligned} \mathrm{d}\theta &= -\pi \wedge \omega - b\omega^F \wedge \theta^F, \\ \mathrm{d}\omega &= 0, \\ \mathrm{d}\pi &= -(-bb^F\omega^F) \wedge \theta - (-b_{zz}w^F - b^F p + b(b^F)_z w)\omega^F \wedge \omega \\ &\quad - b_z\omega^F \wedge \theta^F. \end{aligned}$$

We see $T_{2^F} = -b$, $V_{2^F} = -b_z$ and the other torsion coefficients are zero. ⊘

Next we consider coordinate transformations that leave the normal form (5.29) invariant. We want to prove that these transformations must be of diagonal form, i.e., $x^1 = \phi(y^1)$, $x^2 = \psi(y^2)$, etc. From the analysis of the principal symbol it follows that coordinate transformations that preserve the normal form must preserve the almost product structures on X, E and F. We see that any transformation leaving the normal form (5.29) invariant must be of the form

$$\begin{aligned} x &= \phi(y, y^F), \\ u &= S(y, y^F)v, \quad f = T(y, y^F)g. \end{aligned} \tag{5.31}$$

with ϕ a $\mathbb{D}$-valued function and S, T two 2×2-matrices. The matrices S, T should satisfy either a) $S, T \in D^*$ or b) $S, T \in LD^*$. This follows from the requirement that the base variables are preserved (i.e., no contact transformations), the equation should remain linear in the dependent variable and the almost product structures must be preserved. We plug the transformations (5.31) into the formal form (5.29). If we are in case a), then the result is

$$\begin{aligned} g &= T^{-1}\left(\frac{\partial y}{\partial x^F}\frac{\partial(Sv)}{\partial y} + \frac{\partial y^F}{\partial x^F}\frac{\partial(Sv)}{\partial y^F} + bS^F v^F\right) \\ &= \frac{\partial y}{\partial x^F}T^{-1}S\frac{\partial v}{\partial y} + \frac{\partial y^F}{\partial x^F}T^{-1}S\frac{\partial v}{\partial y^F} + \frac{\partial y}{\partial x^F}T^{-1}\frac{\partial S}{\partial y}v + \frac{\partial y^F}{\partial x^F}T^{-1}\frac{\partial S}{\partial y^F}v + bT^{-1}S^F v^F. \end{aligned}$$

The first term shows that $\partial y/\partial x^F = 0$. The third term automatically vanishes and the fourth term implies that $\partial S/\partial y^F = 0$. We conclude that $x = \phi(y)$ and $S(y, y^F)$ are both hyperbolic holomorphic.

If we are in case *b*), then a similar analysis shows that $\partial y/\partial x = 0$ and $\partial S/\partial y = 0$. In both cases we see that the normal form introduces an almost product structure on the base manifold and the bundles E, F. The transformations in case b) correspond to transformations that interchange the two characteristic systems.

Chapter 6

Structure theory for second order equations

In this chapter we define an adapted coframing for second order scalar equations. The construction of this adapted coframing is similar to the construction of adapted coframings for first order systems in the previous section. Instead of deriving the results again, we will refer to Gardner and Kamran [38] for the main results.

We will use the adapted coframing to define the Monge-Ampère invariant. We also describe the relation between our theory (which is the theory of Gardner and Kamran) and the theory by Juráš [44, 45]. Finally, we give a counterexample to a proposition in [38] about the relations between the class of the characteristic systems and the number of invariants of the characteristic systems.

6.1 Contact geometry of second order equations

The structure theory for second order equations can be done in a way similar to that of first order systems. We will use some existing theory and work with differential forms. The structure theory derived below is only valid for the contact geometry of the system. If one is interested in the point geometry, one has to add some restrictions to the structure group involved and one can not always make all the normalizations that are done below. We assume the second order equation is hyperbolic. Just as for first order systems the theory is almost identical for elliptic systems, but for convenience we only do the hyperbolic case. The parabolic systems have very different structure theory, see Bryant and Griffiths [17] for example.

Let M be the 7-dimensional equation manifold associated to a hyperbolic second order scalar partial differential equation. On M we have the pullback of the contact ideal I and the pair (M, I) completely describes the geometry. The dual distribution $\mathcal{V} = I^{\perp}$ defines a Vessiot system $(M, \mathcal{V})$. The derived system $I^{(1)}$ is 1-dimensional and is spanned by a contact form θ^0.

Theorem 6.1.1 (Proposition 5.5 in [38]). *We can choose generators $\theta^0, \theta^1, \theta^2$ for I such that the following structure equations hold:*

$$\begin{aligned}
\mathrm{d}\theta^0 &\equiv -\theta^1 \wedge \omega^1 - \theta^2 \wedge \omega^2 \mod \theta^0,\\
\mathrm{d}\theta^1 &\equiv -\pi^1 \wedge \omega^1 + T^1_{3^F}\pi^2 \wedge \theta^2 \mod \theta^0, \theta^1,\\
\mathrm{d}\theta^2 &\equiv -\pi^2 \wedge \omega^2 + T^2_{3^F}\pi^1 \wedge \theta^1 \mod \theta^0, \theta^2.
\end{aligned}$$

Note that the last two equations can be written in the more compact form[1]

$$\mathrm{d}\theta \equiv -\pi \wedge \omega + T_{3^F}\pi^F \wedge \theta^F \mod \theta^0, \theta.$$

Here we have written

$$\theta = (\theta^1, \theta^2)^T, \quad \omega = (\omega^1, \omega^2)^T, \quad \pi = (\pi^1, \pi^2)^T \in \mathbb{D} \otimes \Omega(M).$$

From the structure equations it is not difficult to see that $I^{(1)} = \mathrm{span}(\theta^0)$,

$$C(I^{(1)}) = \mathrm{span}\big(\theta^0, \theta^1, \theta^2, \omega^1, \omega^2\big)$$

and $I \subset C(I^{(1)})$. Hence from the theorem of Vessiot (Theorem 4.1.2) it follows that any system with the structure equations from Theorem 6.1.1 is locally equivalent to a second order scalar equation. A closer look at the structure equations implies that the second order equation is hyperbolic.

We also want structure equations for ω and π. The exterior derivative of ω and π can be written modulo θ, ω, π as

$$\begin{aligned}
\mathrm{d}\omega &\equiv U_{2^F,3^F}\omega^F \wedge \pi^F + U_{2^F}\omega^F \wedge \theta^F + U_{3^F}\pi^F \wedge \theta^F,\\
\mathrm{d}\pi &\equiv V_{2^F,3^F}\omega^F \wedge \pi^F + V_{2^F}\omega^F \wedge \theta^F + V_{3^F}\pi^F \wedge \theta^F.
\end{aligned} \tag{6.1}$$

We will show that $U_{2^F,3^F} = 0$. Calculate $\mathrm{d}^2\theta$ modulo $\theta^0, \theta, \theta^F, \omega$. We have

$$0 \equiv \mathrm{d}^2\theta \equiv \pi \wedge \mathrm{d}\omega \equiv \pi \wedge U_{2^F,3^F}\omega^F \wedge \pi^F.$$

Hence the term $U_{2^F,3^F}$ vanishes. A similar calculation of $\mathrm{d}^2\theta$ modulo $\theta^0, \theta, \theta^F$ and π yields $V_{2^F,3^F} = 0$.

Lemma 6.1.2 (T lemma). *Assume the structure equations from Theorem 6.1.1 and equation* (6.1). *If $T^j_3 = 0$, then $U^j_{3^F} = V^j_{3^F} = 0$, for $j = 1, 2$.*

Proof. The proof of the lemma is similar to the proof of the first order T lemma (5.2.8). The main difference that we have to work modulo the additional term θ^0. Assume $T^1_{3^F} = 0$. We calculate $\mathrm{d}^2\theta$ modulo $\theta^0, \theta^1, \omega^1$ and $\theta^2 \wedge \omega^2$.

$$\mathrm{d}^2\theta^1 \equiv -\mathrm{d}\pi^1 \wedge \omega^1 + \pi^1 \wedge \mathrm{d}\omega^1 \equiv \pi^1 \wedge (U^1_{3^F}\pi^2 \wedge \theta^2).$$

Hence $U^1_{3^F} = 0$. A similar calculation modulo $\theta^0, \theta^1, \pi^1, \theta^2 \wedge \omega^2$ yields $V^1_{3^F} = 0$. Calculating $\mathrm{d}^2\theta^2$ modulo similar terms gives $U^2_{3^F} = V^2_{3^F} = 0$. □

[1] Modulo θ^0 means that we work modulo $(1+h)\theta^0$ and $(1-h)\theta^0$.

Proposition 6.1.3. *By redefining $\omega^1, \omega^2, \pi^1, \pi^2$ we can arrange that*

$$\begin{aligned} \mathrm{d}\omega^1 &\equiv \mathrm{d}\pi^1 \equiv 0 \mod \theta^0, \theta^1, \omega^1, \pi^1, \\ \mathrm{d}\omega^2 &\equiv \mathrm{d}\pi^2 \equiv 0 \mod \theta^0, \theta^2, \omega^2, \pi^2. \end{aligned} \tag{6.2}$$

Proof. It is clear from the above that we can always write

$$\mathrm{d}\omega^1 \equiv U^1_{2^F}\omega^2 \wedge \theta^2 + U^1_{3^F}\pi^2 \wedge \theta^2 \mod \theta^0, \theta^1, \omega^1, \pi^1.$$

The coefficient $U^1_{2^F}$ can be absorbed by adding multiples of θ^0 to ω^1. If $T^1_{3^F} = 0$, then $U^1_{3^F} = 0$ by Lemma 6.1.2. Otherwise we can absorb $U^1_{3^F}$ by adding multiples of θ^1. The forms ω^2, π^1 and π^2 can be treated in the same manner. $\square$

We can write $\mathrm{d}\omega \equiv -\pi \wedge (S_{1^F}\theta^F + S_{2^F}\omega^F + S_{3^F}\pi^F)$. Calculate $\mathrm{d}^2\theta^0$ modulo $\theta^0, \omega^1, \omega^2$. We find

$$\begin{aligned} 0 = \mathrm{d}^2\theta^0 &\equiv \theta^1 \wedge \mathrm{d}\omega^1 + \theta^2 \wedge \mathrm{d}\omega^2 \\ &\equiv \theta^1 \wedge \pi^1 \wedge S^1_{3^F}\pi^2 + \theta^2 \wedge \pi^2 \wedge S^2_{3^F}\pi^1. \end{aligned}$$

Hence $S_{3^F} = 0$.

Definition 6.1.4. A coframing for a hyperbolic second order scalar equation is called *adapted* if it has the structure equations

$$\begin{aligned} \mathrm{d}\theta^0 &\equiv -\theta^1 \wedge \omega^1 - \theta^2 \wedge \omega^2 \mod \theta^0 \\ \mathrm{d}\theta &\equiv -\pi \wedge \omega + T_{3^F}\pi^F \wedge \theta^F \mod \theta^0, \theta \\ \mathrm{d}\omega &\equiv -\pi \wedge \sigma \mod \theta^0, \theta, \omega, \\ \mathrm{d}\pi &\equiv 0 \mod \theta^0, \theta, \omega, \pi. \end{aligned} \tag{6.3}$$

with $\sigma = S_{1^F}\theta^F + S_{2^F}\omega^F$. Theorem 6.1.1 and Proposition 6.1.3 together imply that every second order equation has an adapted coframing. $\ominus$

The structure group G that leaves invariant this adapted coframing depends on the value of T_{3^F}. If $T_{3^F} \in \mathbb{D}^*$, then the structure group is given by the set of all matrices of the form

$$\begin{pmatrix} c_0 & 0 & 0 & 0 \\ c_\theta & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & e & ac^{-1} \end{pmatrix} \subset \mathrm{GL}(7, \mathbb{R}), \tag{6.4}$$

with $a, c \in D^*$, $e \in D$, $c_0 \in \mathbb{R}^*$, $c_\theta \in \mathbb{D}$, $c_0 = ac$ and $c_\theta = c_0(c^F)^{-1}T_{3^F}e^F$. If $T_{3^F} = 0$, then the structure group is larger and given by the set of all matrices of the form

$$\begin{pmatrix} c_0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & ac^{-1} \end{pmatrix}$$

with $c_0 = ac$.

Remark 6.1.5. There is a canonical foliation present defined by $\mathrm{d}\theta^0 = \mathrm{d}\theta = \mathrm{d}\omega = 0$. The quotient manifold is a first order contact manifold. This makes the contact geometry of second order systems much more rigid than for first order systems. For first orders systems under contact geometry we have the freedom of switching the role of ω and π. ⊘

6.1.1 Generality of solutions of a second order equation

For hyperbolic equations the characteristic Cauchy problem is well-posed. The general solution of a hyperbolic second order equation depends on two functions of one variable. In terms of initial conditions one can think of these functions as prescribing the value of the function and the value of the first order derivative along a non-characteristic curve. For analytic equations one can prove this using the Cartan-Kähler theorem, see the theorem below.

Theorem 6.1.6. *Let X be an integral element for an analytic hyperbolic second order scalar partial differential equation. Then the general solution to the equation with tangent space equal to the integral element depends on two functions of one variable.*

Proof. Solutions to the equation are integral manifolds of the contact system I spanned by θ^0, θ^1 and θ^2. The structure equations are given by

$$\mathrm{d}\begin{pmatrix}\theta^0\\ \theta^1\\ \theta^2\end{pmatrix} \equiv \begin{pmatrix}0 & 0\\ \pi^1 & 0\\ 0 & \pi^2\end{pmatrix} \wedge \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \quad \bmod I.$$

The Cartan characters are $s_1 = 2, s_2 = 0$. The dimension of the first prolongation is 2, hence the system is in involution. The statement of the theorem follows from the Cartan-Kähler theorem. □

6.1.2 Monge-Ampère invariants

The structure group of the adapted coframing from Definition 6.1.4 depends on the values of T_{3F}, S_{1F} and S_{2F}. The structure group may even vary from point to point. To continue our analysis we are forced to make a choice of one of the different branches that are determined by the invariants. But first we concentrate on some invariants that are defined in all cases.

Suppose we are given an adapted coframing as in Definition 6.1.4. The structure equations for ω can be written as

$$\mathrm{d}\omega \equiv -\pi \wedge (S_{1F}\theta^F + S_{2F}\omega^F) \quad \bmod \theta^0, \theta, \omega.$$

By looking at $\mathrm{d}^2\theta^0$ modulo $\theta^0, \theta^1 \wedge \omega^1, \theta^2 \wedge \omega^2$ we find that $S_{1F} = 0$ and $T_{3F} = (S_{2F})^F$. Let us calculate the action of the structure group (6.4) on T_{3F}. Let $(\tilde{\theta}, \tilde{\omega}, \tilde{\pi})^T = g^{-1}(\theta, \omega, \pi)^T$ Then

$$\begin{aligned}
\mathrm{d}\tilde{\theta} &= -a^{-1}(\mathrm{d}a) \wedge \tilde{\theta} + a^{-1}\mathrm{d}\theta \\
&\equiv a^{-1}(-\pi \wedge \omega + T_{3F}\pi^F \wedge \theta^F) \\
&\equiv -\tilde{\pi} \wedge \tilde{\omega} + a^{-1}(a^F)^2(c^F)^{-1}T_{3F}\tilde{\pi}^F \wedge \tilde{\theta}^F \quad \bmod \tilde{\theta}.
\end{aligned}$$

Hence $\widetilde{T}_{3^F} = a^{-1}(a^F)^2(c^F)^{-1}T_{3^F}$. Under the action of the remaining structure group we have

$$\begin{aligned} T_{3^F} &\mapsto \frac{(a^F)^2}{ac^F}T_{3^F} = \frac{(a^F)^3c}{c_0^2}T_{3^F}, \\ S_{2^F} &\mapsto \frac{ac^F}{c^2}S_{2^F} = \frac{a^3c^F}{c_0^2}S_{2^F}. \end{aligned} \tag{6.5}$$

Hence $T_{3^F} = -(S_{2^F})^F$ is a relative contact invariant. Later we will see that the vanishing of this invariant is a necessary and sufficient condition for the system to be equivalent to a Monge-Ampère equation (this is Theorem 5.3 in Gardner and Kamran [38]).

6.1.3 Analysis of T

The values of the invariants $T^j_{3^F}$ determine the classes of the characteristic systems $M^1 = \operatorname{span}(\theta^0, \theta^1)$ and $M^2 = \operatorname{span}(\theta^0, \theta^2)$.

Proposition 6.1.7 (Proposition 5.6 in Gardner and Kamran [38]). *We have* $\operatorname{class}(M^1) = 7$ *or* $\operatorname{class}(M^1) = 6$ *in the case* $T^1_{3^F} \neq 0$ *or* $T^1_{3^F} = 0$*, respectively. The same is true for* M^2 *and* $T^2_{3^F}$.

Depending on the classes of M^1 and M^2 we can further reduce the structure group.

Proposition 6.1.8. *Suppose* $\operatorname{class}(M^1) = \operatorname{class}(M^2) = 7$*, i.e.,* $T_{3^F} \in \mathbb{D}^*$*. We can normalize* T_{3^F} *to either* $(1, 1)^T$ *or* $(1, -1)^T$*. The structure group reduces to a 3-dimensional group of the form*

$$\begin{pmatrix} \pm\phi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (c_\theta)^1 & \phi & 0 & 0 & 0 & 0 & 0 \\ (c_\theta)^2 & 0 & \pm\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi & 0 & 0 \\ 0 & 0 & 0 & e_1 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & e_2 & 0 & \pm 1 \end{pmatrix}, \tag{6.6}$$

with $c_\theta = \phi(\pm 1, 1)^T e^F T_{3^F}$*. A transformation of the form*

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{6.7}$$

transforms T_{3^F} *to* $-T_{3^F}$.

Proof. Recall that since T^1_{3F} and T^2_{3F} are non-zero, the structure group is already reduced to the form (6.4). The transformation defined by the matrix (6.7) clearly leaves invariant the adapted coframing and transforms T_{3F} to $-T_{3F}$.

The action of the structure group on T_{3F} is by multiplication with $(a^F)^3 c/c_0^2$. By choosing $c_0 = a_1 a_2$ we can make a transformation $T_{3F} \mapsto (a_2^2, a_1^2)^T T_{3F}$. By a suitable choice of a_1 and a_2 we can arrange that $T^1_{3F} = \pm 1$ and $T^2_{3F} = \pm 1$. With a transformation of the form (6.7) we can then arrange $T_{3F} = (1, \pm 1)^T$. □

In the case $T_{3F} \in \mathbb{D}^*$ we do not arrive at an invariant coframe. We could analyze the invariants of the structure equations for the reduced structure group (6.6). Another method is to prolong the system one time. The first prolongation of the Lie algebra corresponding to the reduced structure group is trivial and hence on the prolonged equation manifold there is an invariant coframing. We conclude that the symmetry group of a second order equation with $T_{3F} \in \mathbb{D}^*$ is finite-dimensional. Since the prolonged manifold has dimension 9, the maximal symmetry group is a 9-dimensional Lie group. Equations with a 9-dimensional symmetry group exist [38, Proposition 5.11].

6.2 Miscellaneous

6.2.1 Juráš

In his dissertation Juráš [44] develops a structure theory for second order scalar equations. The work is a continuation of Anderson and Kamran [4] and the results from his dissertation have been published in Juráš and Anderson [45]. The structure theory developed is a theory in local coordinates, although Juráš shows that some of his constructions are contact invariant. The theory uses the variational bicomplex and hence has to be formulated on an infinite jet bundle. The use of local coordinates (in particular the use of specific independent and dependent coordinates) and the fact that Juráš works on the infinite jet bundle makes it difficult to compare his theory to our theory. Still we can make a connection between both theories. In the remainder of this section we assume the reader is familiar with the work of Juráš (either [44] or [45]).

Let us start with a trivial bundle $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ with coordinates x, y, z on $\mathbb{R}^3$ and projection $(x, y, z) \mapsto (x, y)$. The coordinates x, y will be the independent variables and z will be the dependent variable. The infinite prolongation of $\mathrm{J}^\infty(E)$ is defined as the inverse limit of the finite order jet bundles $\mathrm{J}^k(E)$. On the infinite jet bundle $\mathrm{J}^\infty(E)$ we have coordinates $x, y, z, z_x, z_y, z_{xx}$, etc. We have a projection $\pi^\infty_k : \mathrm{J}^\infty(E) \to \mathrm{J}^k(E)$. To a hyperbolic second order partial differential equation

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0,$$

we can associate the equation manifold $\mathcal{R}^\infty \subset \mathrm{J}^\infty(E)$. The equation manifold is defined by all prolongations of the equation $F = 0$. On the infinite jet bundle $\mathcal{R}^\infty$, Juráš constructs an adapted coframing $\Theta, \sigma, \tau, \xi^1, \xi^2, \ldots, \eta^1, \eta^2$, etc.

Given the equation $F = 0$ we can also construct a finite order adapted coframing. We can take for example two distinct roots (λ, μ) of the characteristic equation

$$\frac{\partial F}{\partial z_{xx}}\lambda^2 - \frac{\partial F}{\partial z_{xy}}\lambda\mu + \frac{\partial F}{\partial z_{yy}}\mu^2 = 0.$$

Then we define $\theta_x = \mathrm{d}z_x - z_{xx}\mathrm{d}x - z_{xy}\mathrm{d}y$, $\theta_y = \mathrm{d}z_y - z_{xy}\mathrm{d}x - z_{yy}\mathrm{d}y$ and

$$\begin{aligned}
\theta^0 &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y,\\
\theta^1 &= \lambda_1\theta_x + \mu_1\theta_y, \quad \theta^2 = \lambda_2\theta_x + \mu_2\theta_y,\\
\omega^1 &= \mu_2\mathrm{d}x + \lambda_2\mathrm{d}y, \quad \omega^2 = \mu_1\mathrm{d}x + \lambda_1\mathrm{d}y,\\
\pi^1 &= \lambda_1\mathrm{d}r + \mu_1\mathrm{d}s, \quad \pi^2 = \lambda_1\mathrm{d}r + \mu_1\mathrm{d}s.
\end{aligned}$$

This coframing is adapted in the sense of Definition 6.1.4. Also see the formulas for adapted coframings in Gardner and Kamran [38, formula 3.32].

Our adapted coframing is defined on a hypersurface $\mathcal{R}^2$ in the second order jet bundle $\mathrm{J}^2(E)$. Under the projection π_2^∞ this adapted coframing is pulled back to a set of linearly independent forms on $\mathcal{R}^\infty$. The pullbacks are 1-forms of adapted order 2 or lower. It is not difficult to see that

$$\begin{aligned}
(\pi_2^\infty)^*\theta^0 &= c_0\rho\Theta,\\
(\pi_2^\infty)^*\omega^1 &= c_1\sigma, \quad (\pi_2^\infty)^*\omega^2 = c_2\tau,\\
(\pi_2^\infty)^*\theta^1 &\equiv a_1\eta_1 \mod \Theta,\\
(\pi_2^\infty)^*\theta^2 &\equiv b_1\xi_1 \mod \Theta,\\
(\pi_2^\infty)^*\pi^1 &\equiv a_2\eta_2 \mod \Theta, \eta_3, \xi_3, \eta_4, \xi_4, \ldots,\\
(\pi_2^\infty)^*\pi^2 &\equiv b_2\xi_2 \mod \Theta, \eta_3, \xi_3, \eta_4, \xi_4, \ldots,
\end{aligned} \tag{6.8}$$

where a_j, b_j, c_j are scalar factors. We will not try to make the correspondence more precise by analyzing the scalar factors. We note that to make the correspondence between our finite order adapted coframing and the coframing by Juráš more precise, we could prolong our coframing to a higher order jet bundle and then make the pullback. One of the main achievements of Juráš is that he succeeded in constructing an adapted coframing for all orders at once.

In Proposition 4.4 of [44] Juráš defined two relative contact invariants M_σ and M_τ. In Theorem 10.4 on page 80 he proves that the vanishing of these two invariants is equivalent to the equation being of Monge-Ampère type. The function M_σ is defined as the coefficient of $\tau \wedge \xi_2$ in $\mathrm{d}\sigma$; the function M_τ as the coefficient of $\sigma \wedge \eta_2$ in $\mathrm{d}\sigma$. From the equations (6.8) we can see that M_σ and M_τ correspond up to a scalar factor to the invariants S^1_{2F} and S^2_{2F}, respectively. We can even see that the transformation properties of these relative invariants under contact transformations are identical. Compare formula (4.62) in [44] with our transformation formula (6.5). Here l corresponds to c_0, m to $1/c^1$ and n to $1/c_2$.

We can make a table of corresponding objects in the theory of Juráš, the theory of Gardner and Kamran [38] and the theory by McKay [51] and this dissertation.

Juráš	Gardner and Kamran	McKay, Eendebak
θ	ω^1	θ^0
η^1, ξ^1	π^2, π^3	θ^1, θ^2
-	U, V	$T^1_{3^F}, T^2_{3^F}$
σ, τ	ω^4, ω^6	ω^1, ω^2
η^2, ξ^2	ω^5, ω^7	π^1, π^2
M_σ, M_τ	-	$S^1_{2^F}, S^2_{2^F}$

The correspondence in this table is not exact, but gives an idea of the different objects. The relations are almost always up to a scalar factor and some action of the structure group. The equality $T_{3^F} = -(S_{2^F})^F$ proven in Section 6.1.2 provides the relation between the invariants U, V of Gardner and Kamran and the invariants M_σ, M_τ of Juráš.

6.2.2 Counterexample

In the article [38] Gardner and Kamran discuss the geometry and characteristic systems of second order scalar equations in the plane. Given an adapted coframing (Definition 6.1.4) the characteristic systems are defined as $M^1 = \mathrm{span}(\theta^0, \theta^1)$, $M^2 = \mathrm{span}(\theta^0, \theta^2)$. Recall that the class of M^1 is defined as the corank of the Cauchy characteristic system of $(M^1)^\perp$. An equation is said to be of *generic type* if $\mathrm{class}(M^1) = \mathrm{class}(M^2) = 7$, see Definition 5.2 in [38]. In the article a normal form for the equations of generic type is constructed. Then Corollary 5.9 in [38] states (without proof) that the equations of generic type have no Riemann invariants (a *Riemann invariant* is an invariant of either one of the Monge systems, see [38, Definition 5.4] or [14, §1.4.1]). This statement is incorrect, a counterexample is given below.

Example 6.2.1 (Counterexample to Corollary 5.9 in [38]). Consider the equation $3rt^3 + 1 = 0$. This equation was suggested by Niky Kamran but the equation already is an example in Goursat [40, Exemple IV, p. 130]. The characteristic systems are given by

$$\mathcal{F} = \mathrm{span}\big(F_1 = D_x - (1/t^2)D_y,\, F_2 = \partial_t - (1/t^2)\partial_s\big),$$
$$\mathcal{G} = \mathrm{span}\big(G_1 = D_x + (1/t^2)D_y,\, G_3 = \partial_t + (1/t^2)\partial_s\big),$$

with $D_x = \partial_x + p\partial_z - (1/(3t^3))\partial_p + s\partial_q$ and $D_y = \partial_y + q\partial_z + s\partial_p + t\partial_q$. The invariants for $\mathcal{F}$ and $\mathcal{G}$ are $I_{\mathcal{F}} = \{s + 1/t, x(s + 1/t) - q\}_{\mathrm{func}}$ and $I_{\mathcal{G}} = \{s - 1/t, x(s - 1/t) - q\}_{\mathrm{func}}$, respectively.

The characteristic system dual to the Pfaffian system M^1 is spanned by $\mathcal{F}$, $\mathcal{G}$ and $G_3 = [G_1, G_2] = (2/t^3)(\partial_y + q\partial_z + (s - 1/t)\partial_p)$. The characteristic system dual to M^2 is spanned by $\mathcal{F}$, $\mathcal{G}$ and $F_3 = [F_1, F_2]$. A direct calculation yields that the Cauchy characteristics of both M^1 and M^2 are zero and hence $\mathrm{class}(M^1) = \mathrm{class}(M^2) = 7$. So even though the equation is of generic type it has Riemann invariants. ⊘

Chapter 7

Systems and equations with non-generic Nijenhuis tensor

In this chapter we will study the first order systems and second order scalar equations that have a non-generic Nijenhuis tensor. In the case of first order systems this will lead directly to the equations for pseudoholomorphic curves. The elliptic first order systems that are equations for pseudoholomorphic curves will be given a contact invariant description in terms of the rank of the Nijenhuis tensor. In the case of second order scalar equations a non-generic Nijenhuis tensor will lead to Monge-Ampère equations. The equations for pseudoholomorphic curves and the Monge-Ampère equations have in common that the projection to the base manifold or to the first order contact bundle, respectively, preserves the complex structure (or hyperbolic structure) on the contact distribution.

7.1 Pseudoholomorphic curves

Let (B, J) be a manifold B with an almost complex structure J. The 2-dimensional submanifolds that have tangent spaces that are invariant under J are called *pseudoholomorphic curves* or J-curves. Pseudoholomorphic curves have been introduced by Gromov [41] and have been used in studying symplectic manifolds.

Locally the condition for a submanifold to define a pseudoholomorphic curve defines an elliptic system of partial differential equations. In particular if B has dimension 4 the system of equations is a determined elliptic system of two equations for two functions of two independent variables. In this section we will study the systems that are contact equivalent to this type of equations.

The equations for pseudoholomorphic curves of an almost complex structure are closed under point transformations, but not under contact transformations. The fact that these equations are not closed under contact transformations was a motivation for McKay to develop his theory of pseudocomplex structures, see McKay [52, pp. 1–2].

Below we give a list of 4 classes of elliptic systems that are closed under contact transformations. We will prove that each class is contained in the next one, but strictly smaller. We also give a geometric characterization of the classes and discuss possible normal forms for equations in the different classes.

i) The elliptic systems that have an integrable almost complex structure. These systems are locally contact equivalent to the Cauchy-Riemann equations, see Section 4.6.5. Locally these systems can be written as the equations for pseudoholomorphic curves for an integrable almost complex structure.

ii) The Darboux integrable elliptic first order systems. In Chapter 8 a classification has been made of these systems under contact transformations.

iii) The class of elliptic first order systems that is closed under contact transformations and includes the equations for pseudoholomorphic curves of almost complex structures. We will prove that this class includes the Darboux integrable systems (and in fact is much larger). On the other hand, we will see that this class does not contain all the elliptic systems. We will give a simple characterization of this class in terms of the Nijenhuis tensor. The author is not aware of a geometric description of this class in the literature.

iv) The general elliptic first order system. This class is more or less by definition closed under contact transformations. McKay uses the term pseudocomplex structures or generalized Cauchy-Riemann equations for this class; Gromov uses the term elliptic system and calls the solutions E-curves in analogy with J-curves.

Another class of equations are the equations for pseudoanalytic functions, see Bers [9]. These equations are the elliptic systems of linear, first order partial differential equations in two unknowns and in two independent variables (but other equivalent definitions also exist). The canonical form for such a system is $\partial w/\partial \bar{z} = aw + b\bar{w}$. The condition that the system is linear, is not invariant under point transformations. It would be interesting if there is a geometric condition on an elliptic system that allows it to be written as a linear system.

7.1.1 Projection to an almost complex structure

Let M be a manifold with an almost complex structure J and $\pi : M \to B$ a projection to a manifold B. We will analyze the projections that intertwine the almost complex structure on M with an almost complex structure on B. For every point $m \in M$ we want to have a map $J^B_m : T_{\pi(m)}B \to T_{\pi(m)}B$ such that

$$T_m\pi \circ J_m = J^B_m \circ T_m\pi. \tag{7.1}$$

Assume that at a point $m \in M$ the map J^B_m exists. The projection $T_m\pi$ is a surjective linear map. It follows that J^B_m must be a linear map and it is unique. From

$$J^B_m \circ J^B_m \circ T_m\pi = J^B \circ T_m\pi \circ J_m = T_m\pi \circ J_m \circ J_m = -T_m\pi$$

it follows that $(J_m^B)^2 = -\mathrm{id}$ and hence J_m^B defines a complex structure on $T_{\pi(m)}B$. The map J_m^B is linear so $J_m^B(0) = 0$ and therefore $J_m(\ker T_m\pi) \subset \ker T_m\pi$. Because J_m is injective it follows $J_m(\ker T_m\pi) = \ker T_m\pi$. Hence the fibers of the projection π are J-invariant. If the fibers have dimension two, the fibers are pseudoholomorphic curves in M. The map $T_m\pi$ induces a linear isomorphism $p_m : T_mM/\ker(T_m\pi) \to T_{\pi(m)}B$. Since $\ker(T_m\pi)$ is J_m-linear, the complex structure J_m induces a complex structure on $T_mM/\ker(T_m\pi)$. Then

$$p_m \circ J_m = J_m^B \circ p_m \tag{7.2}$$

and it follows that $J_m^B = p_m \circ J_m \circ p_m^{-1}$. So if a map J_m^B satisfying (7.1) exists, then it is unique and defines a complex structure on $T_{\pi(m)}B$. If $\ker(T_m\pi)$ is J_m-linear, then we can define J_m^B by (7.2). Hence the condition that J_m^B exists is precisely that $\ker(T_m\pi)$ is J_m-linear.

If the fibers of the projection are J-invariant, then at every point $m \in M$ there is a unique complex structure J_m^B on each tangent space $T_{\pi(m)}B$. We say the almost complex structure on M *projects* to B if J_m^B is independent of the point m in the fiber $\pi^{-1}(b)$, $b = \pi(m)$ for each $b \in B$. If this is the case, then we write J_b^B instead of J_m^B. We say the projection π *intertwines* the almost complex structure J on M with the almost complex structure J^B on B. The analysis above shows that if π intertwines J with an almost complex structure on B, then this almost complex structure is unique.

Suppose $\pi : M \to B$ is a projection that intertwines the almost complex structures J and J^B on M and B, respectively. The Nijenhuis tensor on M and on B is defined in terms of the Lie brackets and the almost complex structures. The projection intertwines the almost complex structures and the Lie brackets as well (see Lemma 1.2.23). Therefore for vectors X, Y in T_mM we have

$$T\pi(N(X,Y)) = N(T\pi(X), T\pi(Y)).$$

It follows that if J is integrable, then J^B is integrable as well. The converse is not true in general.

We specialize to the situation that $(M, \mathcal{V})$ is an elliptic first order system, $\pi : M \to B$ a base projection and J is the almost complex structure for this first order system defined in Section 4.6. We want to study the systems for which the projection π intertwines the almost complex structure J with an almost complex structure on B.

The fact that $\ker(T_m\pi)$ is J-invariant and 2-dimensional together with the fact that the distribution $\mathcal{V}$ is J-invariant, implies that either $\ker(T_m\pi) \subset \mathcal{V}$ or $T_mM = \ker(T_m\pi) \oplus \mathcal{V}_m$. In this section we will study the projections for which $\ker(T_m\pi) \subset \mathcal{V}$, we call these projections the *base projections*. The projections for which $T_mM = \ker(T_m\pi) \oplus \mathcal{V}_m$ are called *transversal projections* and are studied in Chapter 8 on Darboux integrability and in Section 9.4 where generalized Darboux projections are introduced.

Pseudoholomorphic curves. Let B be a 4-dimensional manifold with an almost complex structure J^B. A surface in B is a pseudoholomorphic curve if the tangent space to the surface at every point is a J^B-linear subspace of the tangent space. The space of all 2-dimensional linear subspaces of $\mathrm{Gr}_2(T_bB)$ that are J^B-linear is given by $\mathrm{Gr}_2(T_bB, J_b^B)$. We will write

$\mathrm{Gr}_2(TB, J^B)$ for the bundle over B with fiber above a point b equal to $\mathrm{Gr}_2(T_bB, J^B_b)$. This bundle is an elliptic first order system and is equal to the system of equations for pseudoholomorphic curves for J^B.

If an elliptic first order system $(M, \mathcal{V})$ has a base projection π that intertwines the almost complex structure J on M with an almost complex structure J^B on B, then the system is canonically contact equivalent to the equations for pseudoholomorphic curves of J^B. Recall the proof of the weak Vessiot theorem (4.6.3). For any point $m \in M$ the subspace $T_m\pi(\mathcal{V}_m)$ is a 2-dimensional linear subspace of $T_{\pi(m)}B$. This defines a map $\phi : M \to \mathrm{Gr}_2(TB) : m \mapsto T_m\pi(\mathcal{V}_m)$ and this defines a contact transformation $M \mapsto \phi(M) \subset \mathrm{Gr}_2(TB)$.

Since $\mathcal{V}$ is J-invariant, the projected space $T_m\pi(\mathcal{V}_m)$ is J^B-invariant as well. Hence the solutions of the first order system $(M, \mathcal{V})$ have tangent spaces that are J^B-linear. This proves the projections of solutions of $(M, \mathcal{V})$ are pseudoholomorphic curves for J^B. The image $\phi(M)$ is contained in $\mathrm{Gr}_2(TB, J^B)$ and because the fibers of $\tilde{M} \to B$ and $\mathrm{Gr}_2(TB, J^B) \to M$ are 2-dimensional this shows that locally M is contact equivalent to the equations for pseudoholomorphic curves on B, i.e., $\phi(M)$ is equivalent to $\mathrm{Gr}_2(TB, J^B)$.

Conditions for the existence of a projection. We have seen that the equations for pseudoholomorphic curves correspond to elliptic first order systems with a projection to a base manifold that intertwines almost complex structures. In the paragraphs below we will analyze the existence of such intertwining projections.

We will start with a theorem about these systems that is the beginning of a geometric description of the third class (iii) on page 156.

Theorem 7.1.1 (Theorem 3 in McKay [51]). *Let $(M, \mathcal{V})$ be an elliptic first order system with projection $\pi : M \to B$ to a base manifold B and adapted coframing* (5.20) θ, ω, π. *Then the almost complex structure on M projects to an almost complex structure on M if and only if $T_{\bar{3}} = 0$ and $S_{\bar{2}} = 0$.*

Proof. The almost complex structure on M is given by the complex forms θ, ω and π. Since the fibers of the projection are spanned by the vector fields dual to π the complex structure J^B_m is determined by θ_m and ω_m. The condition that J^B_m is independent of the point in the fiber above the point $\pi(m)$ is equivalent to the following four equations modulo $\theta, \omega, \pi, \bar{\pi}$

$$\mathcal{L}_{\partial_\pi}\theta \equiv 0, \quad \mathcal{L}_{\partial_{\bar{\pi}}}\theta \equiv 0,$$
$$\mathcal{L}_{\partial_\pi}\omega \equiv 0, \quad \mathcal{L}_{\partial_{\bar{\pi}}}\omega \equiv 0.$$

Then from the structure equations (5.20) we can read off easily that the conditions translate to $T_{\bar{3}} = 0$ and $S_{\bar{2}} = 0$. □

The theorem allows us to recognize the systems $(M, \mathcal{V})$ that allow a projection to an almost complex structure. The theorem below gives the geometric characterization of the third class of equations given at the beginning of this chapter.

Theorem 7.1.2. *An elliptic first order system $(M, \mathcal{V})$ is the system of equations for pseudoholomorphic curves of an almost complex manifold (B, J^B) if and only if* $\operatorname{rank}\mathcal{D} = 0$ *or* $\operatorname{rank}\mathcal{D} = 2$.

The base projection from the system to the manifold with almost complex structure is unique if and only if $\operatorname{rank}\mathcal{D} = 2$. *If* $\operatorname{rank}\mathcal{D} = 2$, *then the tangent spaces of the fibers of the projection* $M \to B$ *are equal to* $\mathcal{B}_1$.

Proof. If $\mathcal{D} = 0$, then the invariants $T_{\bar 2}$, $T_{\bar 3}$ are zero for any adapted coframing. The system is contact equivalent to the Cauchy-Riemann equations and we are done. The discussion of the Nijenhuis tensor above shows that all projections project to an integrable almost complex structure. From Example 5.4.3 it follows that multiple projections exist.

If $\operatorname{rank}\mathcal{D} \neq 0$, then $(T_{\bar 2}, T_{\bar 3}) \neq 0$ and the Nijenhuis tensor is not identically zero. The bundle $\mathcal{B}_1$ (defined on page 99) has rank 2. The bundle $\mathcal{B}_1$ is the unique rank 2 bundle $\mathcal{U}$ for which $T_{3^F} = 0$ if we arrange that $\mathcal{U}$ is dual to $\operatorname{span}(\theta, \bar\theta, \omega, \bar\omega)$. Hence from Theorem 7.1.1 it follows that $\mathcal{B}_1$ is the only candidate for a projection that intertwines the almost complex structure on M with an almost complex structure on B.

First we will prove that the condition $\operatorname{rank}\mathcal{D} = 2$ is necessary for an intertwining projection to exist. Suppose we have an adapted coframing for which $\mathcal{B}_1$ is a distinguished subbundle. For the existence of an intertwining projection we need that $S_{3^F} = 0$ ($\mathcal{B}_1$ integrable) and $S_{2^F} = 0$. Since $T_{2^F} \neq 0$ it follows from an elliptic version of Lemma 5.2.8 that $U_{\bar 3} = 0$, $V_{\bar 2} = 0$ and hence $\operatorname{rank}\mathcal{D} = 2$.

We assume that $\operatorname{rank}\mathcal{D} = 2$ and prove that this implies that $\mathcal{B}_1$ is integrable. Choose an adapted framing such that the bundle $\mathcal{B}_1$ is a distinguished bundle for the system, see Definition 5.2.7. Then $\mathcal{B}_1$ is dual to $\operatorname{span}(\theta, \bar\theta, \omega, \bar\omega)$. From the definitions of $\mathcal{B}_1$ and the adapted coframing for a distinguished bundle it follows that $T_{\bar 3} = 0$. The rank conditions on $\mathcal{D}$ imply that $T_{\bar 2} \neq 0$ and $U_{\bar 3} = V_{\bar 3} = 0$. Since we have not proved yet that $\mathcal{B}_1$ is integrable we have structure equations

$$
\begin{aligned}
\mathrm{d}\theta &\equiv -\pi \wedge \omega + T_{\bar 2}\bar\omega \wedge \bar\theta \mod \theta,\\
\mathrm{d}\omega &\equiv -\pi \wedge \sigma + U_{\bar 2}\bar\omega \wedge \bar\theta \mod \theta, \omega,\\
\mathrm{d}\pi &\equiv V_{\bar 2}\bar\omega \wedge \bar\theta \mod \theta, \omega, \pi.
\end{aligned}
$$

From an elliptic version of the first order T lemma (Lemma 5.2.8) it follows that $S_{\bar 3} = S_{\bar 2} = 0$. Hence $\mathcal{B}_1$ is integrable and provides a foliation to a base manifold. Since $T_{\bar 3} = S_{\bar 2} = 0$ the complex structure on $\mathcal{V}$ projects down to the base manifold. □

Example 7.1.3. Consider the elliptic first order system

$$
v_y = u_x + u, \quad v_x = -u_y.
$$

We introduce coordinates $x, y, u, v, p = u_x, q = u_y$ for the equation manifold. The complex characteristic systems are given by

$$
\mathcal{V}_+ = \operatorname{span}\left(D_x - iD_y + ip\partial_q, \partial_p + i\partial_q\right)
$$

with $D_x = \partial_x + p\partial_u - q\partial_v$, $D_y = \partial_y + q\partial_u + (p+u)\partial_v$ and $\mathcal{V}_-$ which is the complex conjugate of $\mathcal{V}_+$. The projection to the base manifold is $\pi : M \to B : (x, y, u, v, p, q) \mapsto (x, y, u, v)$.

The projection π intertwines the almost complex structure on M with the almost complex structure J^B on B given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & u \\ -1 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

with respect to the basis ∂_x, ∂_y, ∂_u, ∂_v. $\oslash$

Lifting an almost complex structure. The equations for pseudoholomorphic curves of an almost complex structure define an elliptic first order system. Here we construct a coframing for the first order system from the almost complex structure on the base manifold.

Let B be a four-dimensional manifold with almost complex structure $J^B : TB \to TB$. We can introduce a complex coframing α, β such that the forms α, β are $(1,0)$-forms with respect to the almost complex structure J^B. The image of the Nijenhuis tensor of J^B has rank at most 2. This follows from the fact that the Nijenhuis tensor is an anti-symmetric bi-J^B-antilinear map $TB \to TB$. We can therefore arrange that

$$\begin{aligned} \mathrm{d}\alpha &\equiv Q\bar{\alpha} \wedge \bar{\beta} \mod \alpha, \beta, \\ \mathrm{d}\beta &\equiv 0 \mod \alpha, \beta. \end{aligned}$$

We can introduce complex coordinates p, q for the fibers of the bundle $\mathrm{Gr}_2(TB) \to B$ by letting p, q correspond to the 2-plane defined by the kernel of $\theta = \alpha - p\beta - q\bar{\beta}$. A surface in B is a pseudoholomorphic curve if the tangent spaces to the surface are J^B-linear. This defines an equation on $\mathrm{Gr}_2(TB)$ and it is not difficult to see that the J^B-linear 2-planes are given precisely by $q = 0$. The equation $q = 0$ defines a submanifold M of $\mathrm{Gr}_2(TB)$ of dimension six. Hence the equations for a surface in B to be a pseudoholomorphic curve correspond to the system $M \subset \mathrm{Gr}_2(TB)$.

Let us introduce a coframing on M. We can write $\mathrm{d}\alpha = Q\bar{\alpha} \wedge \bar{\beta} + \zeta_1 \wedge \alpha + \zeta_2 \wedge \beta$ and $\mathrm{d}\beta = \xi_1 \wedge \alpha + \xi_2 \wedge \beta$. On M we define the complex coframe $\theta = \alpha - p\beta$, $\omega = \beta$, $\pi = \mathrm{d}p - \zeta_2 - p\zeta_1 - p\xi_2 - p^2\xi_1$. The fibers of the projection to B are spanned by ∂_π and $\partial_{\bar{\pi}}$. The structure equations for the coframe are

$$\begin{aligned} \mathrm{d}\theta &\equiv -\pi \wedge \omega + Q\bar{\theta} \wedge \bar{\omega} \mod \theta, \\ \mathrm{d}\omega &\equiv 0 \mod \theta, \omega, \\ \mathrm{d}\pi &\equiv V_{\bar{2}}\bar{\omega} \wedge \bar{\theta} \mod \theta, \omega, \pi. \end{aligned} \tag{7.3}$$

If we compare this to the structure equations (5.20) we see that the complex coframing θ, ω, π defines an adapted coframing for M. If the almost complex structure J^B is non-integrable, then $Q \neq 0$ and then also the almost complex structure on M is non-integrable. In that case the fibers of the projection to B are spanned by the bundle $\mathcal{B}_1$ as described in the section above.

Projections of almost product structures. For six-dimensional manifolds M with an almost product structure K we can also consider projections π to a 4-dimensional manifold B such that the projection intertwines the almost product structure on M with an almost product structure on B. If K_m^B satisfies

$$T_m\pi \circ K_m = K_m^B \circ T_m\pi, \tag{7.4}$$

then K_m^B is a linear map with $K_m^B \circ K_m^B$ equal to the identity on B. The tangent spaces to the fibers of the projection must be K-invariant. From $K_m^B \circ K_m^B = \mathrm{id}$ it does not follow automatically that K_m^B defines a hyperbolic structure on $T_{\pi(m)}B$. It can happen that the operator K_m^B does not have the proper multiplicities for the eigenvalues ± 1. If K_m^B exists, then it is unique. If for all points $m \in M$ the map K_m^B only depends on the fiber and defines an almost product structure, then we say the almost product structure on M *projects* to an almost product structure on B (which is uniquely determined by K).

Example 7.1.4 (Quasi-linear systems). A quasi-linear first order system is a system of the form

$$f_1 p + f_2 q + f_3 r + f_4 t = 0, \quad g_1 p + g_2 q + g_3 r + g_4 t = 0,$$

with f_j, g_j functions of the variables x, y, u, v. If the system is hyperbolic, then this defines a system of equations for hyperbolic pseudoholomorphic curves.

The converse is not automatically true. For example the hyperbolic first order system

$$ps - qr = 0, \quad r = 1$$

is not quasi-linear. But this system defines a system for hyperbolic pseudoholomorphic curves. The Monge systems are

$$\mathcal{V}_+ = \operatorname{span}\left(\partial_x + p\partial_u + \partial_v, \partial_p\right), \quad \mathcal{V}_- = \operatorname{span}\left(\partial_y - s\partial_x, \partial_s\right).$$

The derived bundles of the Monge systems have a well-defined projection to the base manifold with coordinates x, y, u, v. Locally the system is contact equivalent to the first order wave equation. ⊘

7.1.2 The Darboux integrable systems

In Section 8.1.4 we define Darboux integrability of elliptic and hyperbolic exterior differential systems. For elliptic first order systems the definition corresponds to the following. An elliptic first order system $(M, \mathcal{V})$ is Darboux integrable if there exist two holomorphic functions z, w for the almost complex structure on M for which $\operatorname{span}(\mathrm{d}z, \mathrm{d}w)$ has real rank four and the kernel (which has rank two) is transversal to the contact distribution $\mathcal{V}$. This definition is clearly contact invariant since the only structures used in the definition are the contact distribution $\mathcal{V}$ and the almost complex structure on M and these structures are both invariant under contact transformations. Hence the Darboux integrable systems are closed under contact transformations.

For a general manifold M of dimension $2n$ with almost complex structure J the existence of k linear independent holomorphic functions at a point m implies that the rank of the image of the Nijenhuis tensor is at most $2n - 2k$. See for example Muškarov [56, Theorem 2.1, p. 286]. The existence of two holomorphic functions for a first order system then implies that the image of the Nijenhuis tensor has rank at most two. For first order systems this is also showed explicitly in Section 8.1.4. Since the rank of $\mathcal{D}$ is at most two, we conclude from Theorem 7.1.2 that all Darboux integrable elliptic first order systems can be written as the equations for the pseudoholomorphic curves of an almost complex structure.

In Chapter 8 we will make a classification of Darboux integrable elliptic first order systems. From the normal forms of the three different classes (the Cauchy-Riemann equations, the affine case, and the almost complex case, see Section 8.3) we can clearly see that these systems have projections to a base manifold with an almost complex structure and that the projection intertwines the two almost complex structures.

7.1.3 Almost product structures

For the hyperbolic first order systems there is the class of systems that can be written as the equations for integral surfaces for an almost product structure. This class is the hyperbolic equivalent of the third class from the list of classes on page 156.

For the elliptic systems the rank of $\mathcal{D}$ completely determines whether a system can be written as a system for an almost complex structure. For the hyperbolic systems the characteristic systems are not conjugate to each other, so there can be an asymmetry in the system. A consequence is that for hyperbolic systems there are more possibilities for the rank of $\mathcal{D}$ and the rank does not completely determine the possible projections.

Theorem 7.1.5. *Let $(M, \mathcal{V})$ be a hyperbolic first order system. Then we have the following possibilities for the ranks of $\mathcal{D}$ and $\mathcal{D}/\mathcal{V}$:*

- $\operatorname{rank}\mathcal{D} = 3, 4$. *There exist no projections to an almost product manifold.*
- $\operatorname{rank}\mathcal{D} = \operatorname{rank}\mathcal{D}/\mathcal{V} = 2$. *There exists a unique projection to a base manifold that preserves the almost product structures. The tangent spaces to the fibers of this projection are given by $\mathcal{B}_1$.*
- $\operatorname{rank}\mathcal{D} = 2$, $\operatorname{rank}\mathcal{D}/\mathcal{V} = 1$. *This case does not have an elliptic equivalent. There exists no projection to an almost product manifold.*
- $\operatorname{rank}\mathcal{D} = 1$. *The condition* $\operatorname{rank}\mathcal{D} = 1$ *alone is not enough to determine whether there exist projections or not. Within this class we can distinct three different subclasses:*
  - *i) There exist multiple projections for the structure.*
  - *ii) There exists a unique projection.*
  - *iii) There exist no projections.*

All three cases occur, see Example 7.1.7 and Example 7.1.8.

- rank $\mathcal{D} = 0$. *The system is then equivalent (under a contact transformation) to the system $u_y = v_x = 0$ (see Example 4.6.5). This system allows many different projections. The elliptic equivalent of this class is the class of Cauchy-Riemann equations.*

Proof. The proof of the theorem is similar to the proof for the elliptic systems. We only have to point out the differences.

Assume that $\mathcal{U}$ is an integrable distribution such that the projection along the leaves of $\mathcal{U}$ projects the almost complex structure on M. Then from a hyperbolic version of Theorem 7.1.1 it follows $T_{3^F} = S_{2^F} = 0$. Since $\mathcal{U}$ is integrable $S_{3^F} = 0$ as well. From Lemma 5.2.8 it follows that $U_{3^F} = V_{3^F} = 0$. But then both distributions $N(TM_+, TM_+)$ and $N(TM_-, TM_-)$ have rank at most one. For the cases rank $\mathcal{D} = 3, 4$ and rank $\mathcal{D} = 2$, rank $\mathcal{D}/\mathcal{V} = 1$ we have either $\operatorname{rank}(N(TM_+, TM_+)) > 1$ or $\operatorname{rank}(N(TM_-, TM_-)) > 1$. Hence for these cases there exists no projection intertwining the almost product structure.

The condition rank $\mathcal{D} = $ rank $\mathcal{D}/\mathcal{V} = 2$ implies that there is a unique foliation for which there exists an adapted coframing such that $T_{3^F} = 0$ and $T_{2^F} \in \mathbb{D}^*$. The proof is then similar to the proof on page 159. The condition rank $\mathcal{D} = 2$ is not sufficient. For rank $\mathcal{D} = 2$, rank $\mathcal{D}/\mathcal{V} = 1$ there exist no projections and Example 7.1.6 shows such systems exist.

In the case rank $\mathcal{D} = 1$ there are 3 possibilities. The existence of the three cases is proven by the examples. □

Example 7.1.6. This is an example of a system where rank $\mathcal{D} = 2$, but rank $\mathcal{D}/\mathcal{V} = 1$. So the ranks of the bundles are not symmetric with respect to the characteristic systems. Consider the system defined by

$$u_y = (u_x)^2, \quad v_x = u.$$

On the equation manifold with coordinates x, y, u, v, p, s we introduce the framing

$$\begin{aligned} F_1 &= D_x + a^2 \partial_s, & F_2 &= \partial_p, & F_3 &= [F_1, F_2] = -\partial_u - 2p\partial_s, \\ G_1 &= D_y - 2pD_x, & G_2 &= \partial_s, & G_3 &= [G_1, G_2] = -\partial_v. \end{aligned}$$

The characteristic systems are $\mathcal{F} = \operatorname{span}(F_1, F_2)$ and $\mathcal{G} = \operatorname{span}(G_1, G_2)$. The hyperbolic structure J acts on the basis as $J(F_j) = F_j$, $J(G_j) = -G_j$.

The image $\mathcal{D}$ of the Nijenhuis tensor is spanned by $N(G_1, G_3) = 4\partial_v = -4G_3$ and $N(G_2, G_3) = -8\partial_s = -8G_2$. Note that $\mathcal{D}$ is contained in the derived system $\mathcal{G}'$ and that rank $\mathcal{D}/\mathcal{V} = 1$. The bundle $\mathcal{B}_1$ equals $\operatorname{span}(F_1, F_2, G_2)$. ⊘

Example 7.1.7 (Multiple projections for rank $\mathcal{D} = 1$). Consider the first order system

$$u_y = v, \quad v_x = 0. \tag{7.5}$$

The projection onto the coordinates x, y, u, v projects the system onto a base manifold with non-integrable almost product structure. An adapted coframing for the system is

$$\begin{aligned} \theta^1 &= \mathrm{d}u - p\mathrm{d}x - v\mathrm{d}y, \\ \theta^2 &= \mathrm{d}v - s\mathrm{d}y, \\ \omega^1 &= \mathrm{d}x, \quad \pi^1 = \mathrm{d}p, \\ \omega^2 &= \mathrm{d}y, \quad \pi^2 = \mathrm{d}s. \end{aligned}$$

The structure equations for the coframe are

$$\begin{aligned} \mathrm{d}\theta^1 &= -\pi^1 \wedge \omega^1 + \omega^2 \wedge \theta^2, \\ \mathrm{d}\theta^2 &= -\pi^2 \wedge \omega^2, \\ \mathrm{d}\omega^1 &= \mathrm{d}\pi^1 = 0, \\ \mathrm{d}\omega^2 &= \mathrm{d}\pi^2 = 0. \end{aligned} \tag{7.6}$$

On the base manifold we have coordinates x, y, u, v and a basis for the tangent space is ∂_x, ∂_y, ∂_u, ∂_v. With respect to this basis the almost product structure is given by

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2v & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The projected characteristic systems are $\mathcal{W}_+ = \mathrm{span}(\partial_x, \partial_u)$ and $\mathcal{W}_- = \mathrm{span}(\partial_y + v\partial_u, \partial_v)$.

Another projection for this equation is given by the adapted coframing

$$\begin{aligned} \theta^1 &= \mathrm{d}u - p\mathrm{d}x - v\mathrm{d}y, \\ \theta^2 &= \mathrm{d}v - s\mathrm{d}y, \\ \omega^1 &= \mathrm{d}x + \mathrm{d}p, \quad \pi^1 = \mathrm{d}p, \\ \omega^2 &= \mathrm{d}y, \quad \pi^2 = \mathrm{d}s. \end{aligned}$$

The structure equations are the same as in (7.6). This gives a foliation to a base manifold generated by the integrable distribution $\mathrm{span}(\partial_p - \partial_x, \partial_s)$. Hence projection onto the variables $\tilde{x} = x + p$, y, u, v gives another projection to an almost product manifold. ⊘

Example 7.1.8. Some systems with $\operatorname{rank} \mathcal{D} = 1$ have a unique projection and some systems have no projection at all. An example is the system

$$u_y = \phi(x, y)v_y + c(u_x)^2, \quad v_x = 0, \tag{7.7}$$

for a generic enough $\phi(x, y)$. For $c = 1$ this system has no projection to an almost product structure. For $c = 0$ the projection to the base manifold B defined by $(x, y, u, v, p, q) \mapsto (x, y, u, v)$ intertwines the almost product structure on the equation manifold with an almost product structure on the base manifold. This projection is the only base projection of the almost product structure for this system. ⊘

7.2 Monge-Ampère equations

The Monge-Ampère equations are a very special class of scalar second order equations in the plane. Here we study the geometry of these equations and give a list of different characterizations of Monge-Ampère equations.

The history of Monge-Ampère equations starts with a problem posed by Gaspard Monge in an article [55]. In this article he consideres the problem of optimally transporting a pile of dirt from a given location to another location. The condition that the transportation is optimal is defined in terms of a cost function. Monge originally considered the total distance as a cost function. The problem for a general cost function is known as the Monge-Kantorovich transport problem. In Ampère [2] the optimal solution to the problem for the specific cost function $|x - y|^2$ was shown to be equivalent to a solution to an equation of the form

$$(\phi_{xx}\phi_{yy} - \phi_{xy}^2)\rho_T(\phi_x, \phi_y) = \rho_0(x).$$

This is special case of what is today known as a Monge-Ampère equation. The general Monge-Ampère equation has the form

$$Az_{xx} + Bz_{xy} + Cz_{yy} + D + E(z_{xx}z_{yy} - z_{xy}^2) = 0, \tag{7.8}$$

with A, B, C, D, E functions of the first order coordinates x, y, z, p, q. The equation has applications in differential geometry, geometrical optics and optimization theory. The principal symbol of the equation (7.8) is given by

$$\xi \mapsto (A + tE)(\xi_x)^2 + (B - 2sE)\xi_x\xi_y + (C + rE)(\xi_y)^2.$$

The discriminant is equal to $(B - 2sE)^2 - 4(A + tE)(C + rE)$. Restricted to the hypersurface defined by the Monge-Ampère equation this is equal to $B^2 - 4AC$. Hence the equation (7.8) is hyperbolic if and only if $B^2 - 4(AC + DE) > 0$. This condition does not depend on the second order coordinates r, s, t. In the following we will concentrate on hyperbolic Monge-Ampère equations.

7.2.1 Geometry

The Monge-Ampère equations are second order equations, but they have the special property that they can also be formulated as an exterior differential system on the first order jet bundle. Recall that any second order scalar equation defines a Vessiot system (Definition 4.1.1) of dimension 7 with a canonical projection to a first order contact manifold P. For a hyperbolic equation we have two characteristic systems $\mathcal{V}_\pm$ with derived bundles $\mathcal{V}'_\pm$. For each point $m \in M$ the projection π maps the rank 3 distribution $\mathcal{V}'_\pm$ to a two-dimensional linear subspace $\mathcal{W}_\pm(m) \subset T_pP$, $p = \pi(m)$. We will analyze the condition that $\mathcal{W}_\pm(m)$ is independent of the point m in the fiber $\pi^{-1}(p)$. If this condition is satisfied, then the projection π defines rank 2 distributions $\mathcal{W}_\pm$ in P. We say that the bundles $\mathcal{V}'_\pm$ project down to P under the projection π. We will see that the equations for which the characteristic systems project down to P are precisely the equations that are contact equivalent to a Monge-Ampère equation.

Example 7.2.1. Consider the *Sine-Gordon equation* $z_{xy} = \sin(2z)$. An adapted coframing for this second order equation is given by

$$\begin{aligned}
\theta^0 &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\
\theta &= \begin{pmatrix} \mathrm{d}p - r\mathrm{d}x - \sin(2z)\mathrm{d}y \\ \mathrm{d}q - \sin(2z)\mathrm{d}x - t\mathrm{d}y \end{pmatrix}, \\
\omega &= \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}, \\
\pi &= \begin{pmatrix} \mathrm{d}r - 2p\cos(2z)\mathrm{d}y \\ \mathrm{d}t - 2q\cos(2z)\mathrm{d}x \end{pmatrix}.
\end{aligned}$$

The structure equations for this coframe are

$$\begin{aligned}
\mathrm{d}\theta^0 &\equiv -\theta^1 \wedge \omega^1 - \theta^2 \wedge \omega^2 \mod \theta^0, \\
\mathrm{d}\theta &\equiv -\pi \wedge \omega \mod \theta^0, \theta, \\
\mathrm{d}\omega &= 0 \\
\mathrm{d}\pi &\equiv 0 \mod \theta, \omega, \pi.
\end{aligned}$$

The Monge systems are

$$\begin{aligned}
\mathcal{V}_+ &= \operatorname{span}\left(\partial_x + p\partial_z + r\partial_p + \sin(2z)\partial_q + 2q\cos(2z)\partial_t, \partial_r\right), \\
\mathcal{V}_- &= \operatorname{span}\left(\partial_y + q\partial_z + \sin(2z)\partial_p + t\partial_q + 2p\cos(2z)\partial_r, \partial_t\right).
\end{aligned}$$

The projection $\pi : M \to P$ to the first order contact bundle projects the derived bundles of the Monge systems to the two characteristic systems

$$\mathcal{W}_+ = \operatorname{span}\left(\partial_x + p\partial_z + \sin(2z)\partial_q, \partial_p\right), \quad \mathcal{W}_- = \operatorname{span}\left(\partial_y + q\partial_z + \sin(2z)\partial_p, \partial_q\right).$$

On the first order jet bundle we can define

$$\begin{aligned}
\theta &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\
\Psi^1 &= (\mathrm{d}p - \sin(2z)\mathrm{d}y) \wedge \mathrm{d}x, \quad \Psi^2 = (\mathrm{d}q - \sin(2z)\mathrm{d}x) \wedge \mathrm{d}y.
\end{aligned}$$

The integral manifolds of the exterior differential system $\{\theta, \Psi^1, \Psi^2\}_{\mathrm{diff}}$ are graphs of 1-jets of functions $z(x, y)$ that satisfy the Sine-Gordon equation. Indeed, when Ψ^1 is restricted to the graph of the 1-jet of a function $z(x, y)$ then

$$\begin{aligned}
(\mathrm{d}p - \sin(2z)\mathrm{d}y) \wedge \mathrm{d}x &= \left(z_{xx}\mathrm{d}x + z_{xy}\mathrm{d}y - \sin(2z)\mathrm{d}y\right) \wedge \mathrm{d}x \\
&= -\left(z_{xy} - \sin(2z)\right) \mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

Note that $\Psi^2 = \Psi^1 - \mathrm{d}\theta$, so we did not need to include Ψ^2 in the definition of our exterior differential system. ⊘

We will analyze when the systems $\mathcal{V}'_\pm$ project down to the first order contact bundle. The fibers of the projection to the first order bundle are given by $C(\mathcal{V}') = C(\mathcal{V}')_+ \oplus C(\mathcal{V}')_-$, with $C(\mathcal{V}')_\pm = C(\mathcal{V}') \cap \mathcal{V}_\pm$. For a good projection we need

$$[C(\mathcal{V}')_+, \mathcal{V}'_+] \subset \operatorname{span}\left(\mathcal{V}'_+, C(\mathcal{V}')\right), \tag{7.9a}$$
$$[C(\mathcal{V}')_+, \mathcal{V}'_-] \subset \operatorname{span}\left(\mathcal{V}'_-, C(\mathcal{V}')\right), \tag{7.9b}$$
$$[C(\mathcal{V}')_-, \mathcal{V}'_+] \subset \operatorname{span}\left(\mathcal{V}'_+, C(\mathcal{V}')\right), \tag{7.9c}$$
$$[C(\mathcal{V}')_-, \mathcal{V}'_-] \subset \operatorname{span}\left(\mathcal{V}'_-, C(\mathcal{V}')\right). \tag{7.9d}$$

The equations above say precisely that the bundle $C(\mathcal{V}')$ defines a vector pseudosymmetry (see Section 9.4) for the two bundles $\mathcal{V}'_\pm$. Another way to formulate the conditions is the following. Take an adapted coframing (6.3). Then the bundles $\mathcal{V}'_+$ and $\mathcal{V}'_-$ are equal to the distributions dual to $\operatorname{span}(\theta^0, \theta^2, \omega^2, \pi^2)$ and $\operatorname{span}(\theta^0, \theta^1, \omega^1, \pi^1)$, respectively. The conditions that $\mathcal{V}'_\pm$ project down the the first order contact manifold are

$$T^1_{3^F} = U^1_{3^F} = 0, \tag{7.10a}$$
$$S^1_{2^F} = S^1_{1^F} = 0, \tag{7.10b}$$
$$S^2_{2^F} = S^2_{1^F} = 0, \tag{7.10c}$$
$$T^2_{3^F} = U^2_{3^F} = 0, \tag{7.10d}$$

(the sublabels correspond to those of (7.9)).

Define the distributions $\mathcal{M}_\pm$ by $\mathcal{M}_\pm = \mathcal{V}'_\pm \oplus \mathcal{V}_\mp$. Note that these distributions are dual to the Pfaffian systems $M^\pm$ defined on page 151. The class of $\mathcal{M}_\pm$ is defined as the corank of $C(\mathcal{M}_\pm)$. This corresponds with the definition of the class of $M^\pm$. Translation of Proposition 6.1.7 to the language of distributions yields the following lemma.

Lemma 7.2.2. *The rank of* $C(\mathcal{M}_\pm)$ *is equal to* 0 *or* 1. *If* $\operatorname{class}(M^\pm) = 6$, *then* $\operatorname{rank} C(\mathcal{M}_\pm) = 1$ *and* $C(\mathcal{M}_\pm) = C(\mathcal{V}')_\pm$*; if* $\operatorname{class}(M^\pm) = 7$, *then* $C(\mathcal{M}_\pm) = \operatorname{span}(0)$.

Lemma 7.2.3. *Let* $(M, \mathcal{V})$ *be the Vessiot systems of a second order scalar partial differential equation. The following 5 statements are equivalent:*

i) The $\operatorname{class} \mathcal{M}_+ = 6$ *and* $\operatorname{class} \mathcal{M}_- = 6$.

ii) The Nijenhuis tensor on M is identically zero.

iii) The integrable distribution $C(\mathcal{V}')$ *generates a projection of the distributions* $\mathcal{V}'_\pm$. *The conditions* (7.9) *hold.*

iv) Any vector field in $C(\mathcal{V}')_\pm$ *is a symmetry of* $\mathcal{V}'_\pm$ *and any vector field in* $C(\mathcal{V}')_\pm$ *is a pseudosymmetry (see Section 9.1) of* $\mathcal{V}'_\mp$.

v) The rank of $\mathcal{V}''_\pm$ *equals four.*

If the conditions above hold, then $C(\mathcal{M}_\pm) = C(\mathcal{V}')_\pm = C(\mathcal{V}'_\pm)$.

Proof. We do the proof in the hyperbolic setting. The proof for the elliptic case follows by complexifying the tangent space. Choose an adapted coframing θ, ω, π for the system. The first condition is equivalent to $T_{3F} = 0$ by Proposition 6.1.7. The vanishing of the Nijenhuis tensor is equivalent to $T_{3F} = 0$ as well. The condition (iii) is equivalent to $T_{3F} = S_{1F} = S_{2F} = U_{3F} = 0$. The condition (iv) is equivalent to $T_{3F} = S_{1F} = S_{2F} = U_{3F} = V_{3F} = 0$. The equivalence of the first four statements follows from Lemma 6.1.2 and the analysis of S_{1F}, S_{2F} in Section 6.1.2.

For any second order equation rank $\mathcal{V}'_\pm = 3$. The derived bundle $\mathcal{V}''_\pm$ has rank 4 or 5, from the structure equations (6.3) it follows that $\mathcal{V}''_\pm = 4$ is equivalent to the first condition. □

We have studied the conditions for the bundles $\mathcal{V}'_\pm$ to project to the first order contact bundle in quite some detail. Let us describe the geometry on the first order contact bundle a bit more. The distribution $\mathcal{V}''_+ \cap \mathcal{V}''_-$ is invariantly defined on M. For a Monge-Ampère equation the rank is 1, for a generic equation ($T_{3F} \in \mathbb{D}^*$) the rank is three. For a Monge-Ampère equation we can define in a similar way $\mathcal{Z} = \mathcal{W}'_+ \cap \mathcal{W}'_-$ on the first order contact bundle. Under the projection $M \to P$ the bundle $\mathcal{V}''_+ \cap \mathcal{V}''_-$ projects to $\mathcal{Z}$. Recall that the bundle $\mathcal{V}'$ projects to a contact structure $\mathcal{W}$ on P.

Let $(M, \mathcal{V})$ be a Monge-Ampère equation with projection to $(P, \mathcal{W})$. For some Monge-Ampère equations we can choose a contact form ω such that the Reeb vector field R of ω spans $\mathcal{Z}$. A direct calculation shows that all equations of the form $s = \phi(x, y, z)$ have this property. If such a contact form exists then it is unique up to a constant scalar. Since ω is defined up to a constant scalar also the volume form $\Omega = \omega \wedge \mathrm{d}\omega \wedge \mathrm{d}\omega$ is invariantly defined up to a constant scalar. This proves that all contact symmetries of such a Monge-Ampère equation are volume preserving diffeomorphisms up to a constant scalar factor. This property is used in Example 9.3.10 to construct all contact symmetries of the wave equation.

Example 7.2.4 (continuation of Example 7.2.1). The characteristic bundles are

$$\mathcal{W}_+ = \operatorname{span}\left(\partial_x + p\partial_z + \sin(2z)\partial_q, \partial_p\right), \quad \mathcal{W}_- = \operatorname{span}\left(\partial_y + q\partial_z + \sin(2z)\partial_p, \partial_q\right).$$

It is clear that $\mathcal{Z} = \mathcal{W}'_+ \cap \mathcal{W}'_- = \operatorname{span}(\partial_z)$. The vector field ∂_z is the Reeb vector field for the contact form θ^0. In the local coordinates $\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}p \wedge \mathrm{d}p$ is the invariant volume form. ⊘

Example 7.2.5. Not all Monge-Ampère equations have the property that $\mathcal{Z}$ is generated by a Reeb vector field. For example the equation $s = y^2 t$ is a hyperbolic Monge-Ampère equation. The contact form and the characteristic 2-forms are given by

$$\begin{aligned} \theta^0 &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\ \Psi^1 &= \mathrm{d}x \wedge (\mathrm{d}p + y^2\mathrm{d}q), \\ \Psi^2 &= (\mathrm{d}y - y^2\mathrm{d}x) \wedge \mathrm{d}q. \end{aligned}$$

The characteristic systems are

$$\mathcal{W}_+ = \operatorname{span}\big(\partial_x + p\partial_z + y^2(\partial_y + q\partial_z), \partial_p\big), \quad \mathcal{W}_- = \operatorname{span}\big(\partial_y + q\partial_z, \partial_q - y^2\partial_p\big).$$

The distribution $\mathcal{Z}$ is spanned by $Z = \partial_z + 2y\partial_p$. There is no function ϕ such that the Reeb vector field R corresponding to the contact form $\phi\theta^0$ is a multiple of Z. $\oslash$

We have seen that the Monge-Ampère equations $(M, \mathcal{V})$ have a natural projection $M \to P$ to a contact manifold $(P, \mathcal{W})$. For hyperbolic equations the Monge systems $\mathcal{V}_\pm$ project to rank 2 distributions $\mathcal{W}_\pm = T\pi(\mathcal{V}_\pm)$ such that $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_-$. The relation $[\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V}$ implies that $[\mathcal{W}_+, \mathcal{W}_-] \subset \mathcal{W}$. The triple $(P, \mathcal{W}_+, \mathcal{W}_-))$ defines a hyperbolic exterior differential system of class $s = 1$ with as additional condition $[\mathcal{W}_+, \mathcal{W}_-] \subset \mathcal{W}$. Also see Example 5.5.4. We will prove that the converse is also true.

Proposition 7.2.6. *Let $(P, \mathcal{W})$ be a contact manifold of dimension 4 with a splitting $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_-$ such that $[\mathcal{W}_+, \mathcal{W}_-] \subset \mathcal{W}$.*

Then there exists a Monge-Ampère equation $(M, \mathcal{V})$ such that $(P, \mathcal{W})$ is the first order contact manifold for this Monge-Ampère equation and the projection $M \to P$ intertwines the Monge systems $\mathcal{V}_\pm$ with the distributions $\mathcal{W}_\pm$.

Proof. Let Q be the bundle over P for which the fiber Q_p above a point $p \in P$ consists of the 2-dimensional integral elements of the distribution $\mathcal{W}$. In other words, if Z is a base manifold for P, then Q is the second order contact bundle of Z. We define a hypersurface M in Q as follows. A 2-plane $E \subset \mathcal{W}_p$ is in M_p if and only if $\dim(E \cap \mathcal{W}_+) = \dim(E \cap \mathcal{W}_-) = 1$. In other words, M_p consists of the 2-dimensional integral planes of the hyperbolic exterior differential system $(M, \mathcal{W}_+, \mathcal{W}_-)$. Then M is a hypersurface in Q transversal to the projection $Q \to P$ and defines a Vessiot system in two coordinates. This is a Monge-Ampère equation with the required properties. $\square$

Hence an alternative definition of a hyperbolic Monge-Ampère equation is a contact manifold $(P, \mathcal{W})$ with a hyperbolic structure on $\mathcal{W}$ such that $[\mathcal{W}_+, \mathcal{W}_-] \subset \mathcal{W}$.

7.2.2 Equivalent definitions of Monge-Ampère equations

One of the first geometric characterizations of Monge-Ampère equations is due to Vessiot. In Vessiot [68, Chapitre III] he characterizes the linear equations and the Monge-Ampère equations in terms of the derived characteristic systems of the Monge systems. The condition he uses, translated to our notation, is that the Cauchy characteristics of $\mathcal{V}'_\pm \oplus \mathcal{V}_\mp$ are equal to $C(\mathcal{V}')_\pm$ [68, p. 311]. A dual formulation in terms of differential forms is given by Theorem 5.3 in Gardner and Kamran [38]. Gardner and Kamran do not mention the results of Vessiot in their paper.

Below we give a list of different geometric characterizations of the hyperbolic Monge-Ampère equations. The elliptic Monge-Ampère equations have similar descriptions. We have the following equivalent formulations (i) to (iv) of hyperbolic Monge-Ampère equations:

i) A hyperbolic Monge-Ampère equation is a hyperbolic second order scalar equation of the form

$$E(rt - s^2) + Ar + Bs + Ct + D = 0,$$

for functions A, B, C, D, E of the first order variables x, y, z, p, q.

ii) In the analytic setting a Monge-Ampère equation is a second order equation that can locally be transformed by a contact transformation into a quasi-linear equation. Hence the Monge-Ampère equations are the smallest group of equations invariant under contact transformations that include the quasi-linear equations.

iii) A Monge-Ampère equation is a hyperbolic exterior differential system $(P, \mathcal{W}_+, \mathcal{W}_-)$ of class $s = 1$ such that $(P, \mathcal{W})$ is a contact manifold and $[\mathcal{W}_+, \mathcal{W}_-] \subset \mathcal{W}$.

iv) A Monge-Ampère equation is a hyperbolic Vessiot system in two variables $(M, \mathcal{V})$ (see Definition 4.1.1 on page 81) that satisfies one of the following equivalent conditions:

   a) The Cauchy characteristics of $\mathcal{V}'_\pm \oplus \mathcal{V}_\mp$ are equal to $C(\mathcal{V}')_\pm$.
   b) The Cauchy characteristics of $\mathcal{V}'_\pm$ are equal to $C(\mathcal{V}')_\pm$.
   c) The rank of $\mathcal{V}''_\pm$ is equal to 4.
   d) $\mathrm{class}(M^1) = \mathrm{class}(M^2) = 1$.
   e) The Nijenhuis tensor on $C(\mathcal{V}') \times_M \mathcal{V}'$ is identically zero.
   f) The canonical projection to the first order contact manifold projects the characteristic systems $\mathcal{V}_+$ and $\mathcal{V}_-$ onto two characteristic subsystems $\mathcal{W}_+$ and $\mathcal{W}_-$, respectively.

Proof. The equivalence (i) with (iv) has been showed many times in the literature. See for example the appendix on Monge-Ampère equations in Bryant and Griffiths [17, pp. 216–218]. The equivalence to (ii) is proved in Lychagin et al. [49]. For the equivalence of (iii) and (iv) and the different conditions (a)– (f) in (iv) see the discussions in this chapter. □

Chapter 8

Darboux integrability

The method of Darboux to integrate partial differential equations was introduced by Gaston Darboux in a paper [21] in 1870. If an equation is Darboux integrable, then the solutions of equation can be obtained by solving ordinary differential equations (integration of functions).

After the original articles by Darboux the main reference seems to be Goursat [40]. Other texts include Stormark [64, Chapter 11], Forsyth [33] and the modern approach using infinite jet bundles by Anderson and Juráš [4, 44, 45]. A classification of the hyperbolic Goursat equations (Example 4.4.2) is obtained by Vessiot [69, 70]. A modern exposition of the work of Vessiot is given by Stormark [64]. The elliptic and hyperbolic first order systems have been analyzed by McKay [51] and Vassiliou [65, 66], respectively. Both analyze the Darboux integrability property, although they use different methods.

In this chapter we will study some properties of the Darboux integrable equations. In particular we will see that any Darboux integrable equation is a special example of our projection method (pseudosymmetry method), discussed in more detail in Chapter 9. We also give a classification of the hyperbolic Darboux integrable first order systems under contact and point transformations at the first order. At the end of this chapter we make an analysis of the homogeneous Darboux integrable equations. Both Vessiot and Vassiliou develop a structure theory for Darboux integrable equations in terms of invariant Lie algebras. In Chapter 10 we will give a geometric interpretation of this structure theory.

We formulate Darboux integrability for hyperbolic exterior differential systems with some additional properties. Special examples are first order systems and second order scalar equations that are Darboux integrable in the classical sense. In Chapter 10 we will generalize Darboux integrability to more general structures. We say the system is Darboux integrable if each characteristic system has at least 2 invariants that satisfy some transversality conditions to be discussed in Section 8.1.2. The term Darboux integrability is also used if one of the prolongations of system has characteristic systems with 2 or more invariants. If we want to express this kind of Darboux integrability, we will always write this explicitly. See Section 8.1.5 for an example of higher order Darboux integrability.

8.1 Some properties of Darboux integrable systems

In this section we give a description of the method of Darboux and discuss some of the properties of the method.

8.1.1 Base and transversal projections

The projections in this section should not be confused with the base projections of a first order systems or second order equation. The base projections are always generated by an integrable pair of vector fields X, Y such that $X, Y \subset \mathcal{V}$. These base projections are treated in detail in Chapter 7.

The projections in this chapter are transversal to the contact distribution $\mathcal{V}$ in the sense that every projection π gives an injective map $T_x\pi : \mathcal{V}_x \to T_{\pi(x)}B$. We call these projections transversal projections or generalized Darboux projections. In this chapter only Darboux projections will be discussed. In chapters 9 and 11 also other transversal projections will be discussed.

8.1.2 The method of Darboux as a projection

We will give a definition of the method of Darboux for hyperbolic exterior differential systems of arbitrary class s. The method of Darboux was originally introduced for second order partial differential equations. Recall from Section 5.5.1 that the Vessiot systems for second order scalar equations define hyperbolic exterior differential systems of class $s = 3$. For second order partial differential equations our definition corresponds to the original method (although we use different notation).

Definition 8.1.1. Let $(M, \mathcal{V}_+, \mathcal{V}_-)$ be a hyperbolic exterior differential system of class s (Definition 5.5.2) that satisfies $[\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$. The system is *Darboux integrable* if there are two invariants I_+^1, I_+^2 for $\mathcal{V}_+$ and two invariants I_-^1, I_-^2 for $\mathcal{V}_-$ such that the forms $\mathrm{d}I_+^1, \mathrm{d}I_+^2, \mathrm{d}I_-^1, \mathrm{d}I_-^2$ are linearly independent and the kernel of $\mathrm{span}(\mathrm{d}I_+^1, \mathrm{d}I_+^2, \mathrm{d}I_-^1, \mathrm{d}I_-^2)$ (which is a rank s distribution on M) is transversal to $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$. $\ominus$

For a Darboux integrable system with invariants $I_+^1, I_+^2, I_-^1, I_-^2$ we define locally a projection $\pi : M \to B \subset \mathbb{R}^2 \times \mathbb{R}^2$ by $m \mapsto (I_+^1(m), I_+^2(m), I_-^1(m), I_-^2(m))$. For any point in M we can arrange by restriction to some smaller neighborhood that $B = B_1 \times B_2$. This projection is called the *Darboux projection* for the system and the invariants. The conditions in Definition 8.1.1 imply that π is a submersion and for all points $m \in M$ the linear map $T_m\pi : \mathcal{V}_m \to T_{\pi(m)}B$ is a bijection. The characteristic systems $\mathcal{V}_+$ and $\mathcal{V}_-$ project to the tangent spaces of the first and second component of $B \subset \mathbb{R}^2 \times \mathbb{R}^2$, respectively. Hence the projection intertwines the hyperbolic structure on $\mathcal{V}$ with the direct product structure on B.

If the characteristic systems $\mathcal{V}_+$ and $\mathcal{V}_-$ have p and q invariants and the system is Darboux integrable, we say the system is (p, q)-Darboux integrable. This terminology was introduced by Vassiliou [66] in his analysis of Darboux integrable systems.

In the special case of a hyperbolic first order system (which is a hyperbolic exterior differential system of class $s = 2$) the manifold M has dimension 6 and carries a natural hyperbolic structure on the entire tangent space and not only on the distribution $\mathcal{V}$. If the system is Darboux integrable, then any Darboux projection is a special case of the transversal projections of almost product structures discussed in Section 7.1.1.

We regard two Darboux projections as different if the fibers of the projections are different. For a Darboux integrable hyperbolic exterior differential system there can exist multiple Darboux projections. This is for example the case for the wave equation or the Cauchy-Riemann equations. For the existence of multiple Darboux projections it is necessary that at least one of the characteristic systems has more than two invariants. This implies that for most Darboux integrable systems there is a unique Darboux projection.

Solutions. If a hyperbolic exterior differential system is Darboux integrable, then we can construct the general 2-dimensional integral manifold of the system by solving equations and integration. In contrast, the 2-dimensional integral manifolds of a general hyperbolic exterior differential system can only be found by solving partial differential equations.

Assume that $(M, \mathcal{V}_+, \mathcal{V}_-)$ is a hyperbolic exterior differential system and $\pi : M \to B$ is a Darboux projection. First we select a hyperbolic pseudoholomorphic curve for the almost product structure on B. Since the almost product structure on B is integrable we can do this as follows. We can select two curves in B_1 and B_2 and the direct product of these curves gives a surface S in B. The tangent space of this surface is the direct sum of the tangent spaces to the two curves and since the tangent spaces to the curves are contained in the tangent spaces of the components B_1, B_2, the tangent space is a hyperbolic line. Hence S is a hyperbolic pseudoholomorphic curve. Conversely, locally every hyperbolic pseudoholomorphic curve is the direct product of two curves.

The inverse image of S in M is a codimension 2 submanifold $\tilde{S}$ and both $\tilde{\mathcal{V}}_+ = \mathcal{V}_+ \cap T\tilde{S}$ and $\tilde{\mathcal{V}}_{-+} = \mathcal{V}_- \cap T\tilde{S}$ are rank 1 distributions on $\tilde{S}$. Together they form the rank 2 distribution $\tilde{\mathcal{V}} = \mathcal{V} \cap T\tilde{S} = \tilde{\mathcal{V}}_+ \oplus \tilde{\mathcal{V}}_-$. Take any pair of vector fields X, Y such that $X \subset \tilde{\mathcal{V}}_+$ and $Y \subset \tilde{\mathcal{V}}_-$. Since X and Y are vector fields on $\tilde{S}$, the commutator $[X, Y]$ is contained in $T\tilde{S}$. But we also have $[X, Y] \subset [\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V}$. But then $[X, Y] \subset \tilde{\mathcal{V}} = \mathcal{V} \cap T\tilde{S}$. Since $\tilde{\mathcal{V}}$ has rank two this shows that $\tilde{\mathcal{V}}$ is integrable. The leaves of $\tilde{\mathcal{V}}$ are integral manifolds of $\tilde{\mathcal{V}}$ and hence integral manifolds of $\mathcal{V}$ as well. From the hyperbolic pseudoholomorphic curve S we have constructed a family of 2-dimensional integral manifolds of the system. Locally this family depends on a choice of a point in a fiber of the projection; so this family depends on s constants.

The converse is true as well. The tangent space of any 2-dimensional integral manifold U of $(M, \mathcal{V}_+, \mathcal{V}_-)$ is contained in $\mathcal{V}$ and hence U is transversal to the projection. The projection $\pi(U)$ is a 2-dimensional submanifold of B that is a pseudoholomorphic curve for the almost product structure on B. We summarize these remarks in the theorem below.

Theorem 8.1.2. *Let $(M, \mathcal{V}_+, \mathcal{V}_-)$ be a Darboux integrable hyperbolic exterior differential system of class s and let $\pi : M \to B$ be a Darboux projection. Then the projection by π and the lifting of pseudoholomorphic curves gives a one-to-one correspondence between pseudoholomorphic curves of the projected manifold and s-dimensional families of integral manifolds of $(M, \mathcal{V}_+, \mathcal{V}_-)$.*

Remark 8.1.3. There are many different approaches to Darboux integrability. We emphasize the projection and after projection the lifting of solutions on the base manifold. This lifting is an integration procedure.

Juráš and Anderson [45] emphasize the view that Darboux integrability implies the existence of additional equations in involution with the original system. These additional equations are given by relations between the invariants. This is the view of McKay [51, Section 9.2–9.3] as well.

The existence of integrable rank 2 subsystems in ideals dual to the Monge systems is used in Bryant et al. [14, p. 65] and Ivey and Landsberg [43, pp. 217–222]. ⊘

8.1.3 Darboux semi-integrability

A hyperbolic system is called *semi-integrable* by the method of Darboux if at least one of the characteristic systems has 2 invariants I_+^1, I_+^2 such that $\operatorname{span}(\mathrm{d}I_+^1, \mathrm{d}I_+^2)$ has rank two and $\operatorname{span}(\mathrm{d}I_+^1, \mathrm{d}I_+^2)^\perp \cap \mathcal{V}$ has rank two. If a system is semi-integrable by the method of Darboux, then the integral manifolds can be found by solving ordinary differential equations. For example see Bryant et al. [14, p. 70]. We prove this result here a hyperbolic first order system. In the example below we will use a variation of the proof to construct the general solution of a Darboux integrable system that has three invariants for one of the characteristic systems.

Theorem 8.1.4. *For a hyperbolic first order system that is Darboux semi-integrable the initial-value problem on a non-characteristic curve can be solved by solving ordinary differential equations.*

Proof. Let the hyperbolic first order system be defined by a 6-dimensional manifold M with characteristic systems $\mathcal{V}_+$, $\mathcal{V}_-$. Let $\gamma : (a, b) \subset \mathbb{R} \to M$ be an integral curve of $\mathcal{V}$. This means that $\gamma'(t) \in \mathcal{V}$ for all $t \in (a, b)$. The curve γ represents the initial data. The condition that γ is non-characteristic is given by the condition that for all $t \in (a, b)$ the vector $\gamma'(t)$ is not contained in $\mathcal{V}_+$ and not contained in $\mathcal{V}_-$.

Let I_+^1, I_+^2 be invariants of $\mathcal{V}_+$ for which the system is Darboux semi-integrable. We define the projection $\pi : M \to \mathbb{R}^2 : m \mapsto (I_+^1(m), I_+^2(m))$. The composition $\pi \circ \gamma$ is a curve δ in $\mathbb{R}^2$. The condition that γ is non-characteristic implies that δ is an immersion.

Define $\tilde{M}$ to be the inverse image of δ. Then $\tilde{M}$ has dimension five and the distributions $\mathcal{W}_+ = \mathcal{V}_+ \cap T\tilde{M}$ and $\mathcal{W}_- = \mathcal{V}_- \cap T\tilde{M}$ have rank two and one, respectively. The distribution $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_-$ on $\tilde{M}$ has rank three and $\mathcal{W}'$ has rank 4. From the properties of $\mathcal{V}$ it follows that the Cauchy characteristics of $\mathcal{W}$ are 1-dimensional and equal to $C(\mathcal{W}) = \mathcal{W}_-$.

The initial curve γ defines a curve in $\tilde{M}$. The flow of this integral curve by the Cauchy characteristics of $\mathcal{W}$ defines a 2-dimensional surface S in $\tilde{M}$. This is an integral surface of $\mathcal{W}$ since γ is an integral curve of $\mathcal{W}$. It follows that the surface in M defined by the surface S in $\tilde{M}$ is an integral surface of $\mathcal{V}$. □

Example 8.1.5. Consider the hyperbolic system defined by the equations $u_y = 0, v_x = u$. We have coordinates x, y, u, v, p, s on the equation manifold M, with the characteristic systems

$$\mathcal{F} = \mathrm{span}\left(\partial_x + p\partial_u + u\partial_v, \partial_p\right), \quad \mathcal{G} = \mathrm{span}\left(\partial_y + s\partial_v, \partial_s\right).$$

The bundle $\mathcal{G}$ has the three invariants x, u, p. Let π be the projection $M \to B = \mathbb{R}^3 : (x, y, u, v, p, s) \mapsto (x, u, p)$. The bundle $\mathcal{F}$ projects to the distribution

$$T\pi(\mathcal{F}) = \mathrm{span}\left(\partial_x + p\partial_u, \partial_p\right).$$

The most general integral curve of the projected bundle is given by $u = \phi(x)$, $p = \phi'(x)$. The inverse image of this curve is a 4-dimensional manifold $\tilde{M}$ with coordinates x, y, v, s. The distributions $\mathcal{F}$ and $\mathcal{G}$ restrict the two distributions

$$\tilde{\mathcal{F}} = \mathcal{F} \cap T\tilde{M} = \mathrm{span}\left(\partial_x + \phi(x)\partial_v\right), \quad \tilde{\mathcal{G}} = \mathcal{G} \cap T\tilde{M} = \mathrm{span}\left(\partial_y + s\partial_v, \partial_s\right).$$

The bundle $\tilde{\mathcal{F}}$ is equal to $C(\tilde{\mathcal{V}})$; the Cauchy characteristics of $\tilde{\mathcal{V}} = \tilde{\mathcal{F}} \oplus \tilde{\mathcal{G}}$. The invariants of $\tilde{\mathcal{F}}$ are y, $w = v - \int \phi(x)$ and s. Locally we can take the quotient of $\tilde{M}$ by the Cauchy characteristics. The quotient $\tilde{B}$ is 3-dimensional with coordinates y, w, s and the distribution $\tilde{\mathcal{G}}$ projects to the distribution on $\tilde{B}$ given by $\mathrm{span}(\partial_y + s\partial_w, \partial_s)$. Finding 2-dimensional integral manifolds of the original system corresponds to finding integral curves of this distribution on $\tilde{B}$. The most general curve is easily seen to be $w = \psi(y)$, $s = \psi'(y)$. The flow of this curve by the Cauchy characteristics $\tilde{\mathcal{F}}$ is the 2-dimensional manifold parameterized by x, y and $v = w + \int \phi(x)$, $s = \psi'(y)$. Together with $u = \phi(x)$ and $p = \phi'(x)$ this defines the general solution of the system:

$$u = \phi(x), \quad v = \psi(y) + \int^x \phi(x). \qquad \oslash$$

8.1.4 Elliptic Darboux integrability

For elliptic exterior differential systems (in particular the elliptic first order systems and elliptic second order equations) we can define Darboux integrability in a similar way as was done for the hyperbolic exterior differential systems in Section 8.1.2.

Let $(M, \mathcal{V}, J)$ be an elliptic exterior differential system (see Definition 5.5.7). We can define the complex eigenspaces $\mathcal{V}_\pm \subset \mathcal{V} \otimes \mathbb{C}$ of the operator J. Let I be a complex valued function that satisfies $i\mathrm{d}I(X) = (\mathrm{d}I \circ J)(X)$ for all $X \in \mathcal{V}$. Then $(\mathrm{d}I)(X + iJX) = (\mathrm{d}I)(X) + i(\mathrm{d}I)(JX) = (1 + i^2)(\mathrm{d}I) = 0$ and hence $\mathrm{d}I(\mathcal{V}_-) = 0$. The converse is also true. If $\mathrm{d}I$ is an invariant for $\mathcal{V}_-$, then restricted to $\mathcal{V}$ we have $i\mathrm{d}I = \mathrm{d}I \circ J$.

We say an elliptic exterior differential system is *Darboux integrable* if $[\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V}$ and the distribution $\mathcal{V}_-$ has two complex invariants I^1, I^2 such that the projection $M \to \mathbb{C}^2 : m \mapsto (I^1(m), I^2(m))$ has rank 4 and is transversal to $\mathcal{V}$. It then follows automatically that the complex conjugates of I^1 and I^2 are invariants for $\mathcal{V}_+$. The projection intertwines the almost complex structure on $\mathcal{V}$ with the complex structure on $\mathbb{C}^2$. Every holomorphic curve in $\mathbb{C}^2$ corresponds locally to a s-dimensional family of 2-dimensional integral manifolds of $(M, \mathcal{V}, J)$. An equivalent formulation of $[\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V}$ is that $[JX, X] \subset \mathcal{V}$ for all $X \subset \mathcal{V}$.

In the case of a Darboux integrable elliptic first order system $(M, \mathcal{V})$ the Darboux projection is a special case of the transversal projections of almost complex structures discussed in Section 7.1.1. In particular the fibers of the projection have tangent spaces that are invariant for the almost complex structure on M and the fibers are pseudoholomorphic curves. A complex invariant of $\mathcal{V}_-$ is an invariant of $\mathcal{V}'_-$ as well. If we choose an adapted coframing θ, ω, π, then a function I is an invariant of $\mathcal{V}_-$ if and only if $\mathrm{d}I \equiv 0 \mod \theta, \bar\theta, \omega, \pi$. But every invariant of $\mathcal{V}_-$ is an invariant of $\mathcal{V}'_-$ as well and therefore $\mathrm{d}I \equiv 0 \mod \theta, \omega, \pi$. This implies that I is a holomorphic function for the almost complex structure on M.

In his Ph.D. thesis [51] McKay also has made an analysis of the elliptic first order systems. We give a short description of McKay's concept of Darboux integrability in the elliptic case, a summary of McKay [51, paragraph 9.1–9.3]. Consider an elliptic first order system on a manifold M with adapted coframing θ, ω, π. The adapted coframing on the equation manifold induces an almost complex structure on M. Using this almost complex structure we can define the operators ∂ and $\bar\partial$ as $\mathrm{d}f = \partial f + \bar\partial f$, $\partial f \equiv 0 \mod \bar\theta, \bar\omega, \bar\pi$ and $\bar\partial f \equiv 0 \mod \theta, \omega, \pi$. Suppose we have two holomorphic functions $z, w : M \to \mathbb{C}$, i.e., $\bar\partial z = \bar\partial w = 0$. We also assume these functions have linearly independent differentials modulo θ. Then the projection $M \to \mathbb{C}^2 : m \mapsto (z(m), w(m))$ is a Darboux projection.

Let U be an integral surface of M and assume that $\mathrm{d}z$ is linearly independent when restricted to U (we case also treat the case that $\mathrm{d}w$ is linearly independent). Then the image of U under z is locally surjective and there is a relation $w = f(z)$, between z and w. The function f is holomorphic since both z and w are holomorphic and U is an integral surface. Conversely, given a holomorphic map $f : \mathbb{C} \to \mathbb{C}$ we can restrict ourselves to the 4-dimensional submanifold of M defined by the relation $z = f(w)$. The differential equation $\theta = 0$ becomes integrable when restricted to this submanifold, i.e., on the submanifold $\mathrm{d}\theta \equiv 0 \mod \theta, \bar\theta$. The leaves of the integrable Pfaffian system $\theta = 0$ on the submanifold are integral manifolds of the system.

Example 8.1.6. We consider a linear elliptic equation $\partial w/\partial\bar z = b(z, \bar z)\bar w$. An adapted coframing for this equation is given by

$$\begin{aligned} \theta &= \mathrm{d}w - p\mathrm{d}z - b\bar w d\bar z, \\ \omega &= \mathrm{d}z, \\ \pi &= \mathrm{d}p - (b_z\bar w + b\bar q)d\bar z. \end{aligned} \tag{8.1}$$

Here $q = b\bar w$. We will show that a sufficient condition for the equation to be Darboux integrable is

$$\frac{\partial^2 \log b}{\partial z \partial \bar z} = |b|^2. \tag{8.2}$$

By calculating the complex Monge systems for a general function b or by using the coframing (5.30) one can proof that the condition is necessary as well.

We take $\tilde z = z$ and $\tilde p = p - (\partial \log(b)/\partial z)w = p - (b_z/b)w$ as holomorphic functions. From $\mathrm{d}\tilde z = \mathrm{d}z = \omega$ it is clear that $\tilde z$ holomorphic. We assume that equation (8.2) holds. A short calculation gives

$$
\begin{aligned}
\mathrm{d}\tilde{p} &= \mathrm{d}p - \left(\frac{\partial^2 \log b}{\partial^2 z}\right) w\omega - \left(\frac{\partial^2 \log b}{\partial z \partial \bar{z}}\right) w\bar{\omega} - (b_z/b)\mathrm{d}w \\
&= \mathrm{d}p - \left(\frac{\partial^2 \log b}{\partial^2 z}\right) w\omega - |b|^2 w\bar{\omega} - (b_z/b)(\theta + p\omega + b\bar{w}\bar{\omega}) \\
&= \pi - (b_z/b)\theta - \left(\frac{\partial^2 \log b}{\partial^2 z}\right) w\omega - \left(\frac{pb_z}{b}\right)\omega.
\end{aligned}
$$

Thus $\tilde{p}$ is holomorphic and clearly $\mathrm{d}\tilde{z}$ and $\mathrm{d}\tilde{p}$ are linearly independent modulo θ. The existence of holomorphic functions implies that there are relations between the invariants. Indeed, $\mathrm{d}\tilde{z} = \omega$ gives $\tau_2 = 0$ and $\mathrm{d}\tilde{p}$ gives $-(b_z/b)\tau_1 + \tau_3 = 0$.

The system is Darboux integrable and we can use the two holomorphic functions to integrate the equation in the following way. Choose a holomorphic function f and define the curve $\tilde{p} = f(\tilde{z})$ in the space with coordinates $\tilde{z}, \tilde{p}$. The inverse image of this curve under the projection $(z, p, w) \mapsto (\tilde{z}, \tilde{p})$ is a manifold of dimension four with complex coordinates (z, w). The contact form θ restricts on this manifold to the differential form

$$
\Theta = \mathrm{d}w - \left(f + \frac{\partial \log b}{\partial z} w\right)\mathrm{d}z - b\bar{w}\mathrm{d}\bar{z}.
$$

The form Θ is closed modulo $\Theta, \bar{\Theta}$ and therefore we can solve w as a function of z using ordinary integration techniques. This gives a solution to the equation that depends on the arbitrary holomorphic function f and a complex integration constant. $\oslash$

For elliptic first order systems McKay also analyzes when we can have enough holomorphic functions for Darboux integrability. A function f can be used for Darboux's method if $\mathrm{d}f = f_1\theta + f_{\bar{1}}\bar{\theta} + f_2\omega + f_3\pi$. But then

$$
0 = \mathrm{d}^2 f = f_1\bar{\partial}\theta + f_{\bar{1}}\bar{\partial}\bar{\theta} + f_2\bar{\partial}\omega + f_3\bar{\partial}\pi = -f_{\bar{1}}\bar{\pi} \wedge \bar{\omega}.
$$

So we see that in fact f is holomorphic and

$$
\begin{aligned}
0 = \mathrm{d}^2 f &= f_1\bar{\partial}\theta + f_2\bar{\partial}\omega + f_3\bar{\partial}\pi \\
&= (f_1\tau_1 + f_2\tau_2 + f_3\tau_3) \wedge \bar{\theta}.
\end{aligned}
$$

The existence of the holomorphic function f therefore implies a linear relation between the invariants. The coefficients of τ_1, τ_2, τ_3 are the coefficients of the Nijenhuis tensor and Darboux integrability implies that the rank of the image of the Nijenhuis tensor $\mathcal{D}$ is at most two. The transversality condition in Darboux integrability then implies that τ_2 and τ_3 are linearly dependent on τ_1. Using the structure group we can arrange that $\tau_2 = \tau_3 = 0$. McKay then makes a distinction into three cases of Darboux integrability, depending on the type of the invariant $\tau_1 = T_{\bar{2}}\bar{\omega} + T_{\bar{3}}\bar{\pi}$. He defines generic Darboux integrability ($T_{\bar{3}} \neq 0$), non-generic Darboux integrability ($T_{\bar{2}} \neq 0$, $T_{\bar{3}} = 0$) and flat Darboux integrability ($T_{\bar{2}} = T_{\bar{3}} = 0$).

Remark 8.1.7. Darboux integrability implies that there are at least two linear relations between the torsion terms τ_1, τ_2 and τ_3. The converse is not true. There are system that are not Darboux integrable, but that do have two relations between the torsion terms. As an example consider the elliptic first order system $p - s = 2u$, $q + r = -2v$. An adapted complex coframing is given by

$$\begin{aligned}\theta &= \mathrm{d}u - p\mathrm{d}x - q\mathrm{d}y + i\left(\mathrm{d}v - (-q - 2v)\mathrm{d}x - (p - 2u)\mathrm{d}y\right),\\ \omega &= \mathrm{d}x + i\mathrm{d}y,\\ \pi &= \mathrm{d}p - i\mathrm{d}q - 2u\bar{\omega} - \theta.\end{aligned}$$

The structure equations are

$$\begin{aligned}\mathrm{d}\theta &\equiv -\pi \wedge \omega \mod \theta,\\ \mathrm{d}\omega &= 0,\\ \mathrm{d}\pi &\equiv \bar{\omega} \wedge \theta \mod \omega, \pi.\end{aligned}$$

It is clear that $\tau_2 = \tau_3 = 0$ and hence that there are two linear relations. The derived systems of $\mathcal{V}_\pm$ stabilize at dimension 5 and therefore each complex Monge system only has one invariant. The system is not Darboux integrable. ⊘

8.1.5 Higher order Darboux integrability

If the characteristic systems $\mathcal{F}$, $\mathcal{G}$ do not have at least two invariants each, then we cannot apply the method of Darboux directly. It is possible to prolong the system. If the prolonged systems $\mathcal{F}^{(1)}$, $\mathcal{G}^{(1)}$ have enough invariants we can apply the method of Darboux to the prolonged system $M^{(1)}$.

Example 8.1.8. Consider the second order equation $s = zp$. Also see Juráš [44, Example 5 on p. 22] and Goursat [40, Tome II, Exemple V, p. 134]. The characteristic systems are given by

$$\mathcal{F} = \operatorname{span}\bigl(D_x + (pq + z^2 p)\partial_t, \partial_r\bigr), \quad \mathcal{G} = \operatorname{span}\bigl(D_y + (p^2 + zr)\partial_r, \partial_t\bigr),$$

with $D_x = \partial_x + p\partial_z + r\partial_p + zp\partial_q$ and $D_y = \partial_y + q\partial_z + zp\partial_p + t\partial_q$. The bundle $\mathcal{F}$ has invariants y and $q - z^2/2$; the bundle $\mathcal{G}$ only has x as an invariant. The equation is not Darboux integrable on the second order jet bundle. We can prolong the system. We parameterize the integral elements of $\mathcal{V}$ in a neighborhood of $\operatorname{span}(D_x, D_y)$ using two parameters a, b as

$$\operatorname{span}\bigl(D_x + (pq + z^2 p)\partial_t + a\partial_r, D_y + (p^2 + zr)\partial_r + b\partial_t\bigr).$$

One can think of a and b as the third order derivatives z_{xxx} and z_{yyy}, respectively.

The characteristic systems on the prolonged manifold are given by

$$\begin{aligned}\mathcal{F}_1 &= \operatorname{span}\left(D_x + (pq + z^2 p)\partial_t + a\partial_r + (3qzp + tp + z^3 p)\partial_b, \partial_a\right),\\ \mathcal{G}_1 &= \operatorname{span}\left(D_y + (p^2 + zr)\partial_r + b\partial_t + (3rp + az)\partial_a, \partial_b\right).\end{aligned}$$

The invariants at third order are $\{y, q - z^2/2, b + (z^4/4) - z^2 q - zt\}_{\text{func}}$ and $\{x, (2pa - 3t^2)/(2p^2)\}_{\text{func}}$. Each prolonged characteristic system has at least two invariants, hence the equation is Darboux integrable at the third order. ⊘

8.2 Hyperbolic Darboux integrability

A hyperbolic first order system is Darboux integrable if there are at least two functionally independent invariants for each of the characteristic systems and these invariants define a Darboux projection that is transversal to the contact distribution. In terms of differential forms and an adapted coframing this means a pair of $\mathbb{D}$-valued functions I_1, I_2 such that $\mathrm{d}I_j \equiv 0 \mod \theta, \omega, \pi, \theta^F$. The transversality condition is equivalent to the condition that $\mathrm{d}I_1$ and $\mathrm{d}I_2$ are linearly independent modulo θ.

Using such a pair of functions we can define a projection onto an open subset of $\mathbb{R}^4 = \mathbb{D}^2$. The Monge systems project into a direct product of $\mathbb{R}^2 \times \mathbb{R}^2$. We call this a Darboux projection, also see Definition 10.3.6. More on this projection and the construction of solutions of the system can be found in [44, 64, 65, 66].

8.2.1 Relations between invariants

Let θ, ω, π be an adapted coframing (5.10). From the structure equations it follows, just as in the elliptic case, that invariants I must satisfy $\mathrm{d}I = f_1\theta + f_2\omega + f_3\pi$ and $\mathrm{d}^2 I = (f_1\tau_1 + f_2\tau_2 + f_3\tau_3) \wedge \theta^F$ modulo θ, ω, π. The existence of invariants for the system implies the existence of linear relations between the terms τ_1, τ_2 and τ_3 and hence relations between the invariants T, U, V (see equation (5.12)).

Remark 8.2.1. Darboux integrability implies relations between the invariants, but the converse is not true. An example is given by $q = u + v$, $r = u - v$. An adapted coframing is given by

$$\begin{aligned}
\theta^1 &= \mathrm{d}u - p\mathrm{d}x - (u+v)\mathrm{d}y, \\
\theta^2 &= \mathrm{d}v - (u-v)\mathrm{d}x - s\mathrm{d}y, \\
\omega^1 &= \mathrm{d}x, \quad \omega^2 = \mathrm{d}y, \\
\pi^1 &= \mathrm{d}p + (v - u - p)\omega^2 + \theta^1 = \mathrm{d}p + \mathrm{d}u - p\mathrm{d}x - (p+2u)\mathrm{d}y, \\
\pi^2 &= \mathrm{d}s + (s - u - v)\omega^1 - \theta^2.
\end{aligned}$$

The structure equations are

$$\begin{aligned}
\mathrm{d}\theta &= -(\omega - \omega^F) \wedge \theta - \pi \wedge \omega + \omega^F \wedge \theta^F, \\
\mathrm{d}\omega &= 0, \\
\mathrm{d}\pi &= -(\omega - \omega^F) \wedge \theta + \begin{pmatrix} -p + u - v \\ -(v + s + u) \end{pmatrix} \omega \wedge \omega^F + h(\omega + \omega^F) \wedge \pi.
\end{aligned}$$

The invariants U, V are identically zero, but $\mathrm{d}\pi^1 \equiv -\theta^1 \wedge \omega^2 \mod \omega^1, \pi^1$. This implies the hyperbolic structure does not have enough invariants to be Darboux integrable. An example of an elliptic system with this property is given in Example 8.1.7. ⊘

Remark 8.2.2. The fact that an invariant of the distribution dual to $\mathrm{span}(\theta, \theta^F, \omega, \pi)$ is an invariant of the distribution dual to $\mathrm{span}(\theta, \omega, \pi)$ corresponds to the fact that an invariant of the vector field system $\mathcal{V}$ must also be an invariant of the derived system $\mathcal{V}'$. The system $\mathrm{span}(\theta, \omega, \pi)$ is precisely the derived system of $\mathrm{span}(\theta, \theta^F, \omega, \pi)$. ⊘

8.2.2 Classification under contact transformations

In this section we will give a classification of Darboux integrable hyperbolic first order systems under contact transformations. Consider the action (5.13) of the contact structure group on the invariant $T = (T_{2^F}, T_{3^F})$. This action has 4 orbits, but if we also allow a change of characteristic systems there are only 3 orbits to consider.

Assuming that the orbit type of the invariants T_{2^F}, T_{3^F} is locally constant we have to consider three possibilities for the invariants T_{2^F}, T_{3^F}. We have locally either:

- $T = 0$. The flat case.
- $(T^1_{2^F}, T^1_{3^F}) = 0, (T^2_{2^F}, T^2_{3^F}) \neq 0$. The case $(T^1_{2^F}, T^1_{3^F}) \neq 0, (T^2_{2^F}, T^2_{3^F}) = 0$ can be reduced to this case by switching the characteristic systems. The systems in this class are all (2, 3)-Darboux integrable. In principle it would be possible for this case to contain branches with (2, 2)-Darboux integrable equations, but it turns out that this is not the case. If $T^1 = 0$, then $U^1 = V^1 = 0$ and hence the distribution dual to $\mathrm{span}(\theta^1, \theta^2, \omega^1, \pi^1)$ has 3 invariants.
- $(T^1_{2^F}, T^1_{3^F}) \neq 0, (T^2_{2^F}, T^2_{3^F}) \neq 0$. This is the most generic situation for Darboux integrable systems. All first order systems in this class are (2, 2)-Darboux integrable.

Flat Darboux integrability

In the case of flat Darboux integrability $\tau_1 = 0$, hence $T_{2^F} = T_{3^F} = 0$. In geometric terms this means that the image of the Nijenhuis tensor is contained in $\mathcal{V}$. From Lemma 4.6.14 it follows that also $\tau_2 = \tau_3 = 0$. Hence the almost product structure is integrable and the equation manifold has a natural direct product structure. In Section 4.6.5 we already showed that all such systems are contact equivalent.

We can choose hyperbolic coordinates w, z, p and an adapted coframing

$$\theta = \mathrm{d}w - p\mathrm{d}z, \quad \omega = \mathrm{d}z, \quad \pi = \mathrm{d}p.$$

The structure equations for this coframing are

$$\mathrm{d}\theta = -\pi \wedge \omega, \quad \mathrm{d}\omega = 0, \quad \mathrm{d}\pi = 0.$$

In local coordinates this system corresponds to the first order wave equation (Example 4.6.5) defined by $u_y = 0$, $v_x = 0$.

Under point geometry not all flat Darboux integrable systems are equivalent. This follows from the fact that the first order wave equation is not invariant under contact transformations and the condition to be flat, i.e., the Nijenhuis tensor vanishes, is invariant under contact transformations. For an example see Example 5.2.10.

Remark 8.2.3. McKay remarks in [51, page 97] that for elliptic first order systems the condition $T_{\bar{2}} = T_{\bar{3}} = 0$ implies the system is equivalent to the Cauchy-Riemann equations. He does not specify whether this is under point or contact geometry. This is a bit confusing since he is working with point geometry in his thesis, but the result is not true under point geometry. A counterexample can be given by taking an elliptic version of Example 5.2.10. Under contact geometry all systems with $T_{\bar{2}} = T_{\bar{3}} = 0$ are indeed equivalent to the Cauchy-Riemann equations. ⊘

(2, 3)-Darboux integrability

Every (2, 3)-Darboux integrable first order system is locally contact equivalent to the system $u_y = v$, $v_x = 0$. We prove this by starting with a 6-dimensional manifold M with a (2, 3)-Darboux integrable hyperbolic structure and introducing suitable coordinates by adjusting the coframing.

We start with an initial $\mathbb{D}$-valued coframing θ, ω, π with the usual structure equations, see (5.10),

$$\mathrm{d}\theta \equiv -\pi \wedge \omega \mod \theta, \theta^F.$$

Since the equation is (2, 3)-Darboux integrable we can assume the system $\theta^2, \omega^2, \pi^2$ is completely integrable. The form θ^2 satisfies $\mathrm{d}\theta^2 \not\equiv 0 \mod \theta^2$ and hence we can introduce coordinates y, v, s and adapt the coframing such that $\theta^2 = \mathrm{d}v - s\mathrm{d}y$, $\omega^2 = \mathrm{d}y$, $\pi^2 = \mathrm{d}s$.

Let us show how the variables y, v and s can be constructed. Let $\mathcal{V}_+$ be the characteristic system dual to $\mathrm{span}(\theta^1, \theta^2, \omega^2, \pi^2)$ and $\mathcal{V}_-$ the system dual to $\mathrm{span}(\theta^1, \theta^2, \omega^1, \pi^1)$. Then $\mathcal{V}'_+$ is integrable and locally we can make the projection $M \to B = M/\mathcal{V}'_+$. The bundle $\mathcal{V}_-$ projects to a bundle $\tilde{\mathcal{V}}_-$. The bundle $\tilde{\mathcal{V}}_-$ is not integrable and hence defines a contact structure. On the quotient space we can choose coordinates y, v, s such that $\tilde{\theta} = \mathrm{d}v - s\mathrm{d}y$, $\tilde{\omega} = \mathrm{d}y$, $\tilde{\pi} = \mathrm{d}s$ is a basis of differential forms and $\tilde{\theta}$ is a form dual to $\tilde{\mathcal{V}}_-$. For $\tilde{\theta}$ we have the structure equation $\mathrm{d}\tilde{\theta} = -\tilde{\pi} \wedge \tilde{\omega}$. The differential forms $\tilde{\theta}$, $\tilde{\omega}$, $\tilde{\pi}$ pull back to semi-basic forms $\theta^2, \omega^2, \pi^2$ on M. It is not difficult to check that θ^2 is a characteristic contact form for the system on M.

The characteristic system $\mathcal{V}_-$ dual to $\mathrm{span}(\theta^1, \theta^2, \omega^1, \pi^1)$ has two invariants. This implies we can adapt the coframing to

$$\begin{aligned}
\mathrm{d}\theta^1 &= -\alpha^1 \wedge \theta^1 - \pi^1 \wedge \omega^1 + T^1_{2^F}\omega^2 \wedge \theta^2 + T^1_{3^F}\pi^2 \wedge \theta^2, \\
\mathrm{d}\omega^1 &\equiv 0 \mod \theta^1, \omega^1, \pi^1, \\
\mathrm{d}\pi^1 &\equiv 0 \mod \theta^1, \omega^1, \pi^1.
\end{aligned}$$

The pair $(T^1_{2^F}, T^1_{3^F})$ is non-zero, since otherwise $\mathcal{V}_-$ would have 3 invariants and then the system is equivalent to the first order wave equation. We can adapt the coframing to make $T^1_{2^F} = 1$, $T^1_{3^F} = 0$.

Let I, J be two functionally independent invariants of $\mathcal{V}_-$. By definition we have $\mathrm{d}I \equiv a_1\theta^1 + b_1\omega^1 + c_1\pi^1$. We calculate $\mathrm{d}^2 I$ and find

$$0 = \mathrm{d}^2 I \equiv a_1 \mathrm{d}\theta^1 \equiv a_1 \omega^2 \wedge \theta^2 \mod \theta^1, \omega^1, \pi^1.$$

Hence $a_1 = 0$. A similar calculation can be made for J and we can conclude that

$$\mathrm{d}I = b_1\omega^1 + c_1\pi^1, \quad \mathrm{d}J = b_2\omega^1 + c_2\pi^1,$$

for certain functions b_1, c_1, b_2, c_2. We can make a transformation of coframing such that in the new coframing $\omega^1 = \mathrm{d}I$ and $\pi^1 = \mathrm{d}J$. By looking at the representations (5.13) we can see that we can make this transformation and keep the normalizations $T_{3^F} = 0$, $U = V = 0$.

The new coframing already has greatly reduced structure equations

$$\begin{aligned}
\mathrm{d}\theta^1 &= -\alpha^1 \wedge \theta^1 - \pi^1 \wedge \omega^1 + T^1_{2^F}\omega^2 \wedge \theta^2, \\
\mathrm{d}\theta^2 &= -\pi^2 \wedge \omega^2, \\
\mathrm{d}\omega^1 &= 0, \quad \mathrm{d}\omega^2 = 0, \\
\mathrm{d}\pi^1 &= 0, \quad \mathrm{d}\pi^2 = 0.
\end{aligned}$$

Since ω^1 and π^1 are exact we can find functions x, p such that $\omega^1 = \mathrm{d}x$ and $\pi^1 = \mathrm{d}p$. Then because $\mathrm{span}(\theta^1, \omega^1, \omega^2)$ is integrable, we can find a function u such that

$$\theta^1 = \mathrm{d}u - A\mathrm{d}x - B\mathrm{d}y,$$

perhaps with a scaling of θ^1. By calculating $\mathrm{d}\theta^1$ and comparing this with our previous expressions for $\mathrm{d}\theta^1$ we find

$$A = p + \phi(x, y), \quad B = v + \psi(x, y).$$

The coframing we have introduced is precisely the coframing associated to $u_y = v + \psi(x, y)$, $v_x = 0$. By replacing u with $u + \int^y \psi$ we arrive at the normal form

$$u_y = v, \quad v_x = 0. \tag{8.3}$$

The general solution of this system was given in Example 8.1.5. An adapted coframing in local coordinates is

$$\begin{aligned}
\theta^1 &= \mathrm{d}u - p\mathrm{d}x - v\mathrm{d}y, \\
\theta^2 &= \mathrm{d}v - s\mathrm{d}y, \\
\omega^1 &= \mathrm{d}x, \quad \omega^2 = \mathrm{d}y \\
\pi^1 &= \mathrm{d}p, \quad \pi^2 = \mathrm{d}s.
\end{aligned} \tag{8.4}$$

The structure equations are

$$\begin{aligned}
\mathrm{d}\theta^1 &= -\pi^1 \wedge \omega^1 + \omega^2 \wedge \theta^2, \\
\mathrm{d}\theta^2 &= -\pi^2 \wedge \omega^2, \\
\mathrm{d}\omega^1 &= 0, \quad \mathrm{d}\omega^2 = 0 \\
\mathrm{d}\pi^1 &= 0, \quad \mathrm{d}\pi^2 = 0.
\end{aligned}$$

(2, 2)-Darboux integrability

The starting point of our analysis is the adapted coframing (5.10). The theory in this section is the hyperbolic equivalent of the theory of McKay in [51, Section 9.5]. An important difference with the theory of McKay is that we do the analysis in the contact geometry setting. In Section 8.2.3 we will also analyze the systems under point geometry.

First we choose a coframing in which $T_{2^F} = 1$, $T_{3^F} = U_{2^F} = U_{3^F} = V_{2^F} = V_{3^F} = 0$. This reduces the bundle and structure group such that on vertical vectors we have (this follows from the covariant derivatives of the invariants, see equations (5.8) on page 126)

$$\alpha - \alpha^F - \gamma^F = 0, \quad \beta = 0, \quad \delta = 0, \quad \zeta = 0.$$

Write $\mu = \gamma - \alpha^F + \alpha$ (hence μ is zero on vertical vectors, i.e., μ is zero modulo θ, θ^F, ω, ω^F, π, π^F). We can arrange

$$\begin{aligned} \mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} &= -\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha^F - \alpha & 0\\ 0 & \epsilon & 2\alpha - \alpha^F\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} \\ &\quad + \begin{pmatrix} -\pi \wedge \omega + \omega^F \wedge \theta^F \\ -\pi \wedge \sigma + \theta \wedge \beta + \omega \wedge \mu \\ -\pi \wedge \mu + \theta \wedge \delta \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} \mu &\equiv 0 \mod \omega^F, \pi, \pi^F, \\ \delta &\equiv 0 \mod \theta^F, \omega^F, \pi, \pi^F, \\ \beta &\equiv 0 \mod \theta^F, \omega, \omega^F, \pi, \pi^F, \\ \sigma &\equiv 0 \mod \theta^F, \omega^F, \pi^F. \end{aligned} \tag{8.5}$$

Our reduced structure group is of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a^F a^{-1} & 0 \\ 0 & e & a^2 (a^F)^{-1} \end{pmatrix} \in \mathrm{GL}(3, \mathbb{D}). \tag{8.6}$$

We calculate the consequences of $\mathrm{d}^2\theta = 0$ modulo θ. We find

$$\begin{aligned} 0 = \mathrm{d}^2\theta &\equiv \alpha \wedge \mathrm{d}\theta - \mathrm{d}\pi \wedge \omega + \pi \wedge \mathrm{d}\omega + \mathrm{d}\omega^F \wedge \theta^F - \omega^F \wedge \mathrm{d}\theta^F \\ &\equiv -\alpha \wedge \pi \wedge \omega + \alpha \wedge \omega^F \wedge \theta^F + (2\alpha - \alpha^F) \wedge \pi \wedge \omega \\ &\quad + \pi \wedge \mu \wedge \omega + \pi \wedge (\alpha - \alpha^F) \wedge \omega + \pi \wedge \omega \wedge \mu \\ &\quad + (\alpha^F - \alpha) \wedge \omega^F \wedge \theta^F - \pi^F \wedge \sigma^F \wedge \theta^F \\ &\quad + \omega^F \wedge \mu^F \wedge \theta^F + \omega^F \wedge \alpha^F \wedge \theta^F \\ &\equiv -\pi^F \wedge \sigma^F \wedge \theta^F - \mu^F \wedge \omega^F \wedge \theta^F \mod \theta. \end{aligned}$$

Hence $\sigma \equiv 0 \mod \theta, \pi, \theta^F$ and $\mu \equiv 0 \mod \theta, \omega, \theta^F$. This implies $\sigma \equiv 0 \mod \theta^F$, i.e., $S_{2^F} = S_{3^F} = 0$, and $\mu = 0$. Calculating $\mathrm{d}^2\omega$ and $\mathrm{d}^2\pi$ modulo θ, ω, π yields

$$\beta \equiv \delta \equiv 0 \mod \theta, \theta^F, \omega, \omega^F, \pi.$$

The fact that the system is Darboux integrable and we have normalized $U = V = 0$ implies that $\mathrm{d}\omega \equiv \mathrm{d}\pi \equiv 0 \mod \omega, \pi$, i.e., the pair ω, π is completely integrable. Using this it follows $\beta \equiv 0 \mod \omega, \pi$ and $\delta \equiv 0 \mod \pi$. Then

$$\begin{aligned} 0 = \mathrm{d}^2\omega &\equiv (\alpha^F - \alpha) \wedge (-\pi \wedge \sigma) - ((2\alpha - \alpha^F) \wedge \pi) \wedge \sigma - \pi \wedge \mathrm{d}\sigma \\ &\equiv -\alpha \wedge \pi \wedge \sigma - \pi \wedge d\sigma \\ &\equiv -\pi \wedge S_{1^F}\mathrm{d}\theta^F \equiv -S_{1^F}\pi \wedge (-\pi^F \wedge \omega^F) \mod \theta, \theta^F, \omega. \end{aligned}$$

Hence $S_{1^F} = 0$ and $\sigma = 0$. Again calculating $\mathrm{d}^2\omega$, but this time modulo θ, ω we find $\beta \equiv 0 \mod \omega$. Write $\beta = B\omega$, $\delta = P\pi$. Then we have

$$\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = -\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha^F - \alpha & 0\\ 0 & \epsilon & 2\alpha - \alpha^F\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} - \begin{pmatrix}\pi \wedge \omega\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\omega^F \wedge \theta^F\\ B\theta \wedge \omega\\ P\theta \wedge \pi\end{pmatrix}.$$

Calculate

$$\begin{aligned} 0 = \mathrm{d}^2\theta &= \Big(-\mathrm{d}\alpha - \big((P + B)\pi + \omega^F\big) \wedge \omega\Big) \wedge \theta, \\ 0 = \mathrm{d}^2\omega &= \Big(\mathrm{d}(\alpha - \alpha^F) + (\mathrm{d}B - B\alpha) \wedge \theta + B\omega^F \wedge \theta^F\Big) \wedge \omega, \\ 0 = \mathrm{d}^2\pi &= \Big(-\mathrm{d}\epsilon + \epsilon \wedge (\alpha^F - 2\alpha) + (\mathrm{d}P - \alpha P) \wedge \theta \\ &\qquad + \omega^F \wedge \theta^F - P(2\alpha - \alpha^F) \wedge \theta\Big) \wedge \pi. \end{aligned} \tag{8.7}$$

From this we deduce (take linear combinations of the first two equations) that $P = -B$ and $\mathrm{d}\alpha - \omega \wedge \omega^F \equiv 0 \mod \theta$. Write $\mathrm{d}\alpha = -\xi \wedge \theta + \omega \wedge \omega^F$. Using this in the second equation of (8.7) we get $\xi \equiv -B^F\omega$ modulo $\theta, \theta^F, \omega^F$. Then

$$\mathrm{d}^2\alpha \equiv -\xi \wedge \pi \wedge \omega + \xi \wedge \omega^F \wedge \theta^F + B^F\omega \wedge \omega^F \wedge \theta^F \mod \theta.$$

Then $\xi \equiv -B^F\omega \mod \theta$ and $\mathrm{d}\alpha = -(B^F\theta + \omega^F) \wedge \omega$. From the second line in equation (8.7) we get $\mathrm{d}B = B\alpha \mod \theta, \omega$. We can absorb the θ term in ξ, so we may assume $\xi = -B^F\omega$. Using $\mathrm{d}^2\alpha$ again we find

$$\mathrm{d}B = B\alpha + B^F\omega. \tag{8.8}$$

From now on we need to distinguish three separate cases: $B = 0$, $B \in \mathbb{D}^*$ and B a non-zero zero-divisor. The reason is we have only 1 invariant left at this stage, the function B. The remaining structure group is 4-dimensional and has the form (8.6). The group acts on B as

$$B \mapsto a^{-1}B.$$

This action has 4 orbits, but two of these orbits are equivalent if we switch the two characteristic systems.

The cases $B = 0$, $B \in \mathbb{D}^*$ and B a non-zero zero-divisor are called the almost product, affine and mixed case, respectively. The naming of the cases is inspired from the work of McKay. The affine case is a Darboux integrable system for which the associated Lie algebra of tangential symmetries, see Chapter 10, is the affine Lie algebra $\mathfrak{aff}(1)$. The affine group appears on the bottom of page 94 in McKay [51], but the author does not know whether McKay was aware of the tangential symmetries. The almost product case is in a certain sense the closest to the flat case where we have a direct product structure. The Lie group associated to the system is the two-dimensional abelian Lie group. The mixed case does not appear in the work of McKay. In the mixed case one characteristic systems behaves as in the affine case and the other as in the almost product case, hence the name.

Affine case. The systems in the first case $B \in \mathbb{D}^*$ are called the *affine* Darboux integrable systems. We can scale B to $1 \in \mathbb{D}$. Then the coframe is uniquely defined up to adding a factor ω to π. We have $\alpha = -\omega$. The structure equations are

$$\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = \begin{pmatrix}\omega\wedge\theta - \pi\wedge\omega + \omega^F\wedge\theta^F\\ \omega^F\wedge\omega + \theta\wedge\omega\\ -\epsilon\wedge\omega + 2\omega\wedge\pi - \omega^F\wedge\pi - \theta\wedge\pi\end{pmatrix}. \tag{8.9}$$

The structure group has dimension two and the Cartan characters are $s_1 = 2$, $s_2 = 0$. The system is in involution, so all analytic affine Darboux integrable equations are contact equivalent. We will continue by finding a normal form for this class. This analysis will also show that all affine Darboux integrable systems are equivalent and every affine Darboux integrable system is contact equivalent to a real analytic affine Darboux integrable system. Since $\mathrm{d}\omega \equiv 0 \mod \omega$ we can find hyperbolic coordinates z and a $\mathbb{D}^*$-valued function w such that $\omega = w\mathrm{d}z$. Substituting this into $\mathrm{d}\omega$ and solving for θ we obtain $\theta = (1/w)(\mathrm{d}w - ww^F\mathrm{d}z^F - q\mathrm{d}z)$ for an unknown function q. Using this in the structure equations for θ we find $\pi = (1/w^2)\mathrm{d}q - (q/w^3 + 1/w)\mathrm{d}w - H\mathrm{d}z$. We make the substitution $q = w(p+w)$ and use z, w, p as coordinates on the equation manifold. We find the following coframing for the equation:

$$\begin{aligned}\theta &= (1/w)\left(\mathrm{d}w - w(p+w)\mathrm{d}z - ww^F\mathrm{d}z^F\right),\\ \omega &= w\mathrm{d}z,\\ \pi &= (1/w)(\mathrm{d}p - F\mathrm{d}z).\end{aligned} \tag{8.10}$$

Here F is an arbitrary function of the variables z, w, p. Conversely, given any function F the above coframe defines an elliptic system on the manifold with coordinates z, w, p.

Assume that we are working with $\omega\wedge\omega^F$ as an independence condition. On any integral surface we can write w, p as functions of z. Since $\theta = \theta^F = 0$ on any integral surface we find $0 = \mathrm{d}\theta = -\pi\wedge\omega = -\mathrm{d}p\wedge\mathrm{d}z = -((\partial p/\partial z^F)/(ww^F))\omega\wedge\omega^F$. The function z is a hyperbolic holomorphic function on the equation manifold. The function p satisfies

$\partial p/\partial z^F = 0$, so p is hyperbolic holomorphic. The function w satisfies the system

$$\frac{\partial w}{\partial z} = w(p + w), \tag{8.11a}$$

$$\frac{\partial w}{\partial z^F} = w w^F. \tag{8.11b}$$

The converse is also true. If w satisfies $\partial w/\partial z^F = w w^F$, then defining p by the equation (8.11a) above gives a hyperbolic holomorphic function p.

Example 8.2.4. The transformations in the variables z, w, p that leave the normal form (8.11) invariant can be easily determined from the structure equations. First notice that if we introduce new coordinates Z, W, P then from the invariance of ω it follows $W\mathrm{d}Z = w\mathrm{d}z$, hence Z is a hyperbolic holomorphic function of z and $\partial W/\partial z^F = 0$. Then $W\mathrm{d}Z = WZ'\mathrm{d}z = w\mathrm{d}z$ implies that $W = w/Z'$. By analyzing the transformation of θ we arrive at the following transformations

$$\begin{aligned} Z &= Z(z), \\ W &= w/Z', \\ P &= \frac{p - Z''/Z'}{Z'}. \end{aligned} \tag{8.12}$$

$\oslash$

Example 8.2.5 (General solution for affine Darboux integrable system). The equation $\partial w/\partial z^F = w w^F$ is the normal form for the class of hyperbolic affine Darboux integrable equations. We give an explicit expression for the general solution of the equation using the method of Darboux.

First we solve the system of equations (8.11) for $p = 0$. We have a system of four equations for the two functions w^1, w^2 in the variables x^1, x^2. The compatibility equations are satisfied (since $p = 0$ is hyperbolic holomorphic) and by ordinary integration we arrive at the most general solution of the equation

$$w = \frac{N}{1 - Nz - (Nz)^F}. \tag{8.13}$$

Here $N \in \mathbb{D}^*$ is a constant and w is defined for $z = (x^1, x^2)^T$ in a suitable neighborhood of $0 \in \mathbb{D}$.

We will use the equations (8.12) to transform the general system into a system for which $p = 0$. We can then use our special solution (8.13) to find the general solution. We assume p is a hyperbolic holomorphic function and want to solve the system (8.11). We can transform the system for this function p into the system in the coordinates (Z, W, P) for which $P = 0$ by solving the equation $p - Z''/Z' = 0$. We find $Z(z) = \int \exp(\int p(z))$. Since $\exp(\int p(z))$ takes values in $\mathbb{D}^*$, the function $Z(z)$ is invertible. If we let W be the most general solution of the system for $P = 0$, given by (8.13), then $W = N/(1 - NZ - (NZ)^F)$. The general solution of (8.11b) is given by

$$w = WZ' = \frac{f'(z)}{1 - f(z) - f(z)^F}. \tag{8.14}$$

Here $f(z)$ is an arbitrary hyperbolic holomorphic function. In terms of the original function p we have $f(z) = N \int \exp(\int p(z))$. $\oslash$

Almost product case. The second is case is $B = 0$. The structure equations are

$$\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = -\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha^F - \alpha & 0\\ 0 & \epsilon & 2\alpha - \alpha^F\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} - \begin{pmatrix}-\pi \wedge \omega + \omega^F \wedge \theta^F\\ 0\\ 0\end{pmatrix}, \tag{8.15}$$

and $\mathrm{d}\alpha = \omega \wedge \omega^F$. Notice that $\mathrm{d}(\alpha + \alpha^F) = 0$, hence we can write $\alpha = \xi + h\eta$ where $\xi = \mathrm{d}t$ and η are real hyperbolic forms and h is the hyperbolic imaginary number with matrix representation $\mathrm{diag}(1, -1)$. By applying a transformation $\theta \mapsto \exp(t)\theta, \pi \mapsto \exp(t)\pi, \epsilon \mapsto \exp(t)\epsilon$ we can eliminate the term $\mathrm{d}t$ from the structure equations. The new structure equations are

$$\begin{aligned}
\mathrm{d}\theta &= -h\eta \wedge \theta - \pi \wedge \omega + \omega^F \wedge \theta^F,\\
\mathrm{d}\omega &= 2h\eta \wedge \omega,\\
\mathrm{d}\pi &= -\epsilon \wedge \omega - 3h\eta \wedge \pi,\\
\mathrm{d}\xi &= 0,\\
\mathrm{d}\eta &= -h\omega \wedge \omega^F.
\end{aligned}$$

The structure equations for ω and η are those of the Lie group $\mathrm{SL}(2, \mathbb{R})$. We can introduce coordinates $z = (z_1, z_2), \phi$ such that

$$\begin{aligned}
\omega &= \frac{e^{h\phi}\mathrm{d}z}{1 - zz^F},\\
\eta &= (1/2)\mathrm{d}\phi + (1/2)\frac{1}{1 - zz^F}(z\mathrm{d}z^F - z^F\mathrm{d}z).
\end{aligned}$$

Write $\Psi = e^{h\phi/2}/\sqrt{1 - zz^F}$. If we make a transformation of the coframe that leaves invariant the normalizations made so far, then we can only scale ω by a purely imaginary hyperbolic number. This means we cannot arrange that $\mathrm{d}\omega = 0$ and $T_{2^F} = 1$ at the same time. The best thing we can do is scale ω such that the new ω is exact and the new invariant T_{2^F} is real hyperbolic.

We apply the transformation $\theta \mapsto \Psi\theta$, $\omega \mapsto \Psi^{-2}\omega$, $\pi \mapsto \Psi^3\pi$, $\epsilon \mapsto \Psi^5\epsilon$. In the new coframing we have

$$\begin{aligned}
\mathrm{d}\theta &= \mathrm{d}z \wedge \left(\pi + \frac{z^F}{1 - zz^F}\theta\right) + \frac{1}{1 - zz^F}\mathrm{d}z^F \wedge \theta^F,\\
\omega &= \mathrm{d}z,\\
\mathrm{d}\pi &= -\tilde{\epsilon} \wedge \omega = \mathrm{d}z \wedge \tilde{\epsilon},
\end{aligned} \tag{8.16}$$

with $\tilde{\epsilon} = \epsilon + (3z^F/(1 - zz^F))\pi$. Then using the relative Poincaré lemma A.2.3 we can write

$$\begin{aligned} \theta &= \mathrm{d}w - A\mathrm{d}z - B\mathrm{d}z^F, \\ \omega &= \mathrm{d}z, \\ \pi &= \mathrm{d}p - C\mathrm{d}z, \end{aligned} \tag{8.17}$$

for functions A, B, C. It follows that $\tilde{\epsilon} = \mathrm{d}C + \kappa\mathrm{d}z$. By substitution of these expressions in the coframing (8.16) we find

$$\begin{aligned} A &= p + \frac{wz^F}{1 - zz^F} + F(z, z^F), \\ B &= \frac{w^F}{1 - zz^F} + G(z, z^F). \end{aligned}$$

The functions F, G cannot be arbitrary but must satisfy the equation

$$-\frac{\partial F}{\partial z^F} + \frac{\partial G}{\partial z} + \frac{z^F G - G^F}{1 - zz^F}. \tag{8.18}$$

The transformations that leave invariant the coframing (8.17) and structure equations (8.16) above are the transformations $z \mapsto z$, $w \mapsto w + \phi(z, z^F)$, $p \mapsto p + \psi(z)$, with ψ a hyperbolic holomorphic function. Under such a transformation the functions F, G transform as

$$\begin{aligned} F &\mapsto F + \phi z^F/H + \psi - \phi_z, \\ G &\mapsto G + \phi^F/H - \phi_{z^F}. \end{aligned}$$

The condition that the new G is identically zero is a determined hyperbolic system for ϕ, which means we can find a ϕ such that $G = 0$. Then if we choose $\phi = 0$ the equation to transform F to zero reduces to $F + \psi = 0$. Since $G = 0$ was transformed to zero, the condition (8.18) guarantees that $\partial F/\partial z^F = 0$, hence we can transform F to zero. We have arrived at the final coframing for our equations

$$\begin{aligned} \theta &= \mathrm{d}w - \left(p + \frac{wz^F}{1 - zz^F}\right)\mathrm{d}z - \frac{w^F}{1 - zz^F}\mathrm{d}z^F, \\ \omega &= \mathrm{d}z, \\ \pi &= \mathrm{d}p - C\mathrm{d}z. \end{aligned} \tag{8.19}$$

All almost product Darboux integrable equations are equivalent under contact transformations.

Mixed case. We expect a class of equations here in the hyperbolic case that has no elliptic equivalent, but it turns out that this class of equations is empty. Recall that our invariant B must satisfy the equation (8.8). If $B \neq 0$ but $B \notin \mathbb{D}^*$, then we can assume $B^1 = 0$, $B^2 \neq 0$. From the structure equation it follows that $\mathrm{d}B^1 = B^1\alpha^1 + B^2\omega^1$, hence $0 = B^2\omega^1$. This would imply $\omega^1 = 0$ which is not possible. Hence there are no equations of mixed type.

8.2.3 Point transformations

If we consider point transformations, then the different equivalence classes under contact transformations will split into several equivalence classes for the point transformations.

McKay [51, Chapter 9] made an analysis of Darboux integrable elliptic first order systems under point transformations. The fact that he works with point transformations follows from the structure group he uses, see pages 36–40. His analysis of the class of "generic Darboux equations" in Section 9.4 is incorrect due to some calculational errors. With his methods we could make the analysis of the different classes complete. However, we can use our classification under contact transformations to arrive at the same results faster.

Generic and non-generic Darboux integrable equations

For the $(2,2)$-Darboux integrable equations the Nijenhuis tensor has an image with rank 2. The bundle $\mathcal{B}_1$ has rank 2. From the adapted coframings (8.9) and (8.15) we can see that for all $(2,2)$-Darboux integrable systems the distribution $\mathcal{B}_1$ is integrable. The fact that $\mathcal{B}_1$ is integrable if the rank of $\mathcal{D}$ is two was already proved for elliptic first order systems in Section 7.1, see the proof of Theorem 7.1.2.

The leaves of $\mathcal{B}_1$ locally define a foliation of the equation manifold M. We can use this foliation to construct a base manifold. For equations under point symmetries we also have a foliation to the base manifold.

Definition 8.2.6. Let $(M, \mathcal{V})$ be a generalized hyperbolic or elliptic first order system with $M \to B$ a projection to a base manifold and assume the system is $(2,2)$-Darboux integrable.

If the distribution defined by the tangent spaces of the projection to the base manifold is equal to the distribution $\mathcal{B}_1$ we call the system *non-generic*. If the distributions are not identical we call the system *generic*. $\ominus$

For the non-generic equations the point transformations and contact transformations are the same: the fibers of the projection to the base manifold are leaves of the invariant distribution $\mathcal{B}_1$ and hence they are invariantly defined.

Lemma 8.2.7. *Suppose we are given an adapted coframing* (5.6) *for a hyperbolic first order system under point geometry. The system is non-generic if and only if* $T_{3^F} \neq 0$.

The non-generic systems

The normal forms for the almost product and the affine Darboux integrable systems that have been given in the previous sections are automatically non-generic. The contact transformations of these non-generic systems are equal to the point symmetries. The symmetry groups are infinite-dimensional and transitive.

The generic systems: almost product case

Using the fact that we can arrange T_{2^F}, U, V to be zero we can restrict to a subbundle on which we have structure equations

$$\begin{aligned} \mathrm{d}\theta &= -\alpha \wedge \theta - \pi \wedge \omega + \pi^F \wedge \theta^F, \\ \mathrm{d}\omega &= -(2\alpha - \alpha^F) \wedge \omega + \pi \wedge \sigma, \\ \mathrm{d}\pi &= -(\alpha^F - \alpha) \wedge \pi, \\ \mathrm{d}\alpha &= \pi \wedge \pi^F. \end{aligned} \tag{8.20}$$

with $\sigma = S_1\theta + S_{1^F}\theta^F + S_2\omega + S_{2^F}\omega^F$. The structure group for this bundle is the group of diagonal matrices with entries $(a, a^2(a^F)^{-1}, a^F a^{-1})$, $a \in \mathbb{D}^*$.

After some calculations we arrive at the following normal form for the adapted coframing.

$$\begin{aligned} \theta &= \mathrm{d}w - \left(\frac{wp^F}{1-|p|^2} - z\right)\mathrm{d}p - \frac{w^F}{1-|p|^2}\mathrm{d}p^F, \\ \omega &= \mathrm{d}z - \phi\mathrm{d}p, \\ \pi &= \mathrm{d}p. \end{aligned} \tag{8.21}$$

Here ϕ is an arbitrary function satisfying the equation

$$\phi_{p^F} + \phi^F\phi_{z^F} + \frac{w^F}{1-|p|^2}\phi_w + \left(\frac{pw^F}{1-|p|^2} - z\right)\phi_{w^F} = 0. \tag{8.22}$$

This conditions follows from $\mathrm{d}^2\omega \equiv 0 \mod \theta, \theta^F, \omega, \omega^F$. By writing down the characteristic systems $\mathcal{F}, \mathcal{G}$ and calculating the derived bundles one can easily check that for any function ϕ satisfying the equation above, the coframe (8.21) defines a (2, 2)-Darboux integrable system. The structure equations are

$$\begin{aligned} \mathrm{d}\theta &= -\pi \wedge \omega + \frac{1}{1-|p|^2}\pi^F \wedge \theta^F, \\ \mathrm{d}\omega &= \phi_w\pi \wedge \theta + \phi_z\pi \wedge \omega + \pi \wedge (\phi_{z^F}\omega^F + \phi_{w^F}\theta^F), \\ \mathrm{d}\pi &= 0. \end{aligned}$$

The Cartan-Kähler theorem guarantees that there is a family of analytic functions ϕ depending on two functions of five variables that satisfy equation (8.22). The group of transformations of the base manifold depends on four functions of four variables. This shows that there is a continuous family of almost product Darboux integrable systems that are not equivalent under point transformations.

The generic systems: affine case

We apply the transformation $\omega \mapsto \pi, \pi \mapsto -\omega + \phi\pi$ to the structure equations (8.9). This leads to the following structure equations

$$\begin{aligned}
\mathrm{d}\theta &= \pi \wedge \theta - \pi \wedge \omega + \pi^F \wedge \theta^F, \\
\mathrm{d}\omega &= (-\theta - \pi^F) \wedge \omega + \pi \wedge (S_1\theta + S_{1^F}\theta^F + S_2\omega + S_{2^F}\omega^F), \\
\mathrm{d}\pi &= \pi^F \wedge \pi + (\theta \wedge \pi).
\end{aligned} \tag{8.23}$$

We can use the local coordinates from equation (8.10) to introduce coordinates for the generic equations. Every generic system has coordinates z, w, p such that

$$\begin{aligned}
\theta &= w^{-1}\big(\mathrm{d}w - w(p+w)\mathrm{d}z - ww^F\mathrm{d}z^F\big), \\
\omega &= \mathrm{d}p + \phi\mathrm{d}z, \\
\pi &= -\mathrm{d}z.
\end{aligned} \tag{8.24}$$

The coframing is not fully adapted, but the contact distribution is given by $\theta = 0$ and the projection to the base manifold is given by $\theta = \omega = 0$. The function ϕ is arbitrary except that we need $\mathrm{d}\theta \equiv \mathrm{d}\omega \equiv 0 \mod \theta, \theta^F, \omega, \omega^F$. This is equivalent to the system

$$\phi_{z^F} - \phi_{p^F}\phi^F + \phi_w(ww^F) + \phi_{w^F}\left(w^F(p^F + w^F)\right).$$

Just as for the almost product case there is a continuous family of non-equivalent affine Darboux integrable systems under point geometry.

The generic systems: flat and (2,3) case

Under point transformations the (2, 3)-integrable systems and the flat systems split into different classes. We will only give some examples, but will not try to give a complete classification. An example of two flat systems that are contact equivalent, but not equivalent under point transformations are the system from Example 5.2.10 and the first order wave equation (4.15).

Example 8.2.8. Consider the first order system

$$u_y = v_x, \quad u_x + (v_x)^2 = 0. \tag{8.25}$$

The system is hyperbolic near points $v_x \neq 0$. We use coordinates $x, y, u, v, a = u_y, b = v_y$. An adapted coframing is given by

$$\begin{aligned}
\theta^1 &= -a^{-1}(\mathrm{d}u + (a^2/2)\mathrm{d}x - a\mathrm{d}y), \\
\theta^2 &= a^{-1}\,(\mathrm{d}v - a\mathrm{d}x - b\mathrm{d}y) - a^{-1}\theta^1, \\
\omega^1 &= \mathrm{d}x - \frac{1}{a}\mathrm{d}y, \quad \omega^2 = \frac{1}{a}\mathrm{d}y, \\
\pi^1 &= \mathrm{d}a, \quad \pi^2 = \frac{1}{a}\mathrm{d}a + \mathrm{d}b.
\end{aligned} \tag{8.26}$$

The structure equations for this coframe are

$$\begin{aligned}
&\mathrm{d}\theta^1 \equiv -\pi^1 \wedge \omega^1 \mod \theta^1, \\
&\mathrm{d}\theta^2 \equiv -\pi^2 \wedge \omega^2 + \frac{1}{a^2}\pi^1 \wedge \theta^1 \mod \theta^2, \\
&\mathrm{d}\omega^1 \equiv \pi^1 \wedge a^{-1}\omega^2 \mod \theta^1, \omega^1, \quad \mathrm{d}\omega^2 \equiv 0 \mod \theta^2, \omega^2, \\
&\mathrm{d}\pi^1 \equiv 0 \mod \theta^1, \omega^1, \pi^1, \quad \mathrm{d}\pi^2 \equiv 0 \mod \theta^2, \omega^2, \pi^2.
\end{aligned} \tag{8.27}$$

A calculation of the derived systems shows quickly that the equation is (2, 3)-Darboux integrable. However, the invariant $T^2_{3F} = 1/(2a^2)$ is non-zero and hence the system is not equivalent to the system (8.3) (which has $T_{3F} = 0$) by a point transformation. ⊘

8.3 Elliptic Darboux integrability

The structure in the elliptic setting is very similar to the structure in the hyperbolic setting. A difference is that in the elliptic setting we can make all kinds of connections to the theory of Riemann surfaces. The author is not aware of an equivalent to this in the hyperbolic setting.

Theorem 8.3.1. *Under contact equivalence there are 3 different classes of Darboux integrable elliptic systems. These classes are:*

Cauchy-Riemann equations. *These are characterized by the fact that the almost complex structure is integrable.*

Affine case. *We can introduce a complex coframing with structure equations*

$$\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = \begin{pmatrix}\omega\wedge\theta - \pi\wedge\omega + \bar\omega\wedge\bar\theta\\ \bar\omega\wedge\omega + \theta\wedge\omega\\ -\epsilon\wedge\omega + 2\omega\wedge\pi - \bar\omega\wedge\pi - \theta\wedge\pi\end{pmatrix}. \tag{8.28}$$

Almost complex case. *We can introduce a complex coframing with structure equations*

$$\mathrm{d}\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} = -\begin{pmatrix}\alpha & 0 & 0\\ 0 & \bar\alpha-\alpha & 0\\ 0 & \epsilon & 2\alpha-\bar\alpha\end{pmatrix}\wedge\begin{pmatrix}\theta\\ \omega\\ \pi\end{pmatrix} - \begin{pmatrix}-\pi\wedge\omega + \bar\omega\wedge\bar\theta\\ 0\\ 0\end{pmatrix}. \tag{8.29}$$

The structure group consists of the matrices in $\mathrm{GL}(3,\mathbb{C})$ *of the form*

$$\begin{pmatrix}a & 0 & 0\\ 0 & \bar a/a & 0\\ e & 0 & a^2/\bar a\end{pmatrix}.$$

Proof. The proof is identical to the analysis of hyperbolic Darboux integrable systems. The (2, 3)-Darboux integrable systems cannot occur in the elliptic setting since the complex invariants of the characteristic systems always occur in pairs.

The classification also follows from the classification of McKay [51, Section 9.4–9.8] of the Darboux integrable systems under point symmetries and taking together the classes that are equivalent under contact transformations. □

Example 8.3.2 (Affine Darboux integrability). See McKay [51, pp. 92–94] and McKay [52, p. 37]. Take a Riemann surface with local coordinate z. The canonical bundle for the surface is the bundle of $(1,0)$-forms. Take a local section ξ for this bundle. In local coordinates the section ξ can be written as $\xi = w\mathrm{d}z$. We can write down two invariant equations for the section ξ:

$$\mathrm{d}\xi = 0, \tag{8.30}$$

$$\mathrm{d}\xi = \bar{\xi} \wedge \xi. \tag{8.31}$$

The first equation can be written in local coordinates as $\partial w/\partial \bar{z} = 0$. The solutions of the equation are precisely the holomorphic $(1,0)$-forms. The second equation can be written in local coordinates as

$$\frac{\partial w}{\partial \bar{z}} = |w|^2.$$

This is precisely the normal form for the affine Darboux integrable elliptic first order systems. The geometrical significance of the solutions to this equation is unclear to the author.

We can find some special solutions by writing $w = u + iv$. Then we find the coupled system of equations

$$u_x - v_y = 2(u^2 + v^2), \quad u_y + v_x = 0.$$

The second equations implies that there is a potential s such that $(u, v) = (s_x, -s_y)$. The first equation then becomes

$$s_{xx} + s_{yy} = 2(s_x^2 + s_y^2).$$

Solutions with $s_y = 0$ are given by $s = (1/2)\ln(x + c)$. Hence $u = 1/(2(x + c))$, $v = 0$. ⊘

Example 8.3.3 (Almost complex Darboux integrability). The almost complex Darboux integrable system has the normal form $\partial w/\partial \bar{z} = \bar{w}/(1-|z|^2)$. On the unit disk in the complex plane we have a metric and volume form

$$\mathrm{d}s^2 = \frac{|\mathrm{d}z|^2}{(1-|z|^2)}, \quad \Omega = \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{1-|z|^2}.$$

Write $\xi = w\mathrm{d}x$. Then the equation $\partial w/\partial \bar{z} = \bar{w}/(1-|z|^2)$ can be written as

$$\mathrm{d}\xi = \bar{\xi} \wedge \omega,$$

with $\omega = \mathrm{d}z/(1-|z|^2)$. Again there is the question if the equation above has a geometric interpretation in the context of complex Riemann surfaces. ⊘

8.4 Homogeneous Darboux integrable systems

We make an analysis of the homogeneous (2, 2)-Darboux integrable elliptic systems. Here we mean by *homogeneous* that the group of point symmetries acting on the system is transitive and hence at least 6-dimensional. We will see that the symmetry groups are finite-dimensional for the generic equations and hence we can write the equation manifolds as $M = G/H$, where G is the symmetry group of the equation and H the isotropy subgroup. We also show that the flat case and the (2, 3)-case are homogeneous, i.e., they have a transitive symmetry group.

McKay also made an analysis of this type of systems [51, Section 9.4.7]. The analysis of McKay is sketchy, although most of the gaps can be filled. Furthermore some of the classes he finds are empty, and finally he misses the classes of generic affine equations.

8.4.1 Flat case

Under contact transformations the flat case has a large symmetry group. The adapted coframing θ, ω, π has structure equations

$$\begin{aligned}
\mathrm{d}\theta &= -\alpha \wedge \theta - \pi \wedge \omega, \\
\mathrm{d}\omega &= -\beta \wedge \theta - \gamma \wedge \omega - \eta \wedge \pi, \\
\mathrm{d}\pi &= -\delta \wedge \theta - \epsilon \wedge \omega - (\alpha - \gamma) \wedge \pi,
\end{aligned}$$

with structure group G given by matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & c & g \\ d & e & f \end{pmatrix} \subset \mathrm{GL}(3, \mathbb{C}),$$

with $\Delta = cf - eg = a \neq 0$.

The Cartan characters are $s_1 = 6$, $s_2 = 4$, $s_3 = 2$ and the dimension of the first prolongation is 20. Hence the system is in involution and there is a transitive symmetry group depending on 2 real functions of 3 real variables.

There are several different orbits of the point symmetry group in the flat equations. The orbit corresponding to the normal form $u_x = v_y, u_y = -v_x$ has a symmetry group depending on 4 real functions of 2 variables.

Example 8.4.1 (Cauchy-Riemann equations in local coordinates). In local coordinates an adapted coframing for the Cauchy-Riemann equations is given by

$$\theta = \mathrm{d}w - p\mathrm{d}z, \quad \omega = \mathrm{d}z, \quad \pi = \mathrm{d}p.$$

The contact transformations near the identity are all defined by a new foliation

$$\theta = \mathrm{d}w - p\mathrm{d}z, \quad \omega = \mathrm{d}z + \phi\mathrm{d}p, \quad \pi = \mathrm{d}p.$$

The structure equations for this new coframing are

$$\begin{aligned}
\mathrm{d}\theta &= -\pi \wedge \omega, \\
\mathrm{d}\omega &= -\pi \wedge \mathrm{d}\phi \\
&= -\pi \wedge (\phi_{\bar{w}}\bar{\theta} + (\phi_{\bar{z}} + \bar{p}\phi_{\bar{w}})\bar{\omega} + (\phi_{\bar{p}} - \bar{\phi}\phi_{\bar{z}} - \bar{\phi}\bar{p}\phi_{\bar{w}})\bar{\pi}), \\
\mathrm{d}\pi &= 0.
\end{aligned}$$

The condition that the new coframing defines a foliation to a base manifold is $\mathrm{d}\theta \equiv \mathrm{d}\omega \equiv 0 \mod \theta, \bar{\theta}, \omega, \bar{\omega}$. This is precisely equivalent to the condition that $\phi_{\bar{p}} - \bar{\phi}(\phi_{\bar{z}} + \bar{p}\phi_{\bar{w}})$. For the new equation to be equivalent to the Cauchy-Riemann equations under point transformations we need that ϕ is a holomorphic function of z, w, p. $\oslash$

8.4.2 (2, 3)-Darboux integrable systems

The author has not analyzed the contact or point symmetry groups of these systems in detail. Under contact symmetries the symmetry group is transitive and at least 6-dimensional. This follows from the coframing (8.4), which has constant structure coefficients and hence has a transitive symmetry group. The full symmetry group is much larger (and probably not finite-dimensional)

8.4.3 Affine systems

The structure equations for the generic affine elliptic systems are given by

$$\begin{aligned}
\mathrm{d}\theta &= \pi \wedge \theta - \pi \wedge \omega + \bar{\pi} \wedge \bar{\theta}, \\
\mathrm{d}\omega &= (-\theta - \bar{\pi}) \wedge \omega + \pi \wedge (S_1\theta + S_{\bar{1}}\bar{\theta} + S_2\omega + S_{\bar{2}}\bar{\omega}), \\
\mathrm{d}\pi &= \bar{\pi} \wedge \pi + (\theta \wedge \pi).
\end{aligned} \tag{8.32}$$

These equations are the elliptic equivalent of (8.23). The structure group has been reduced to the identity. If our system is to be homogeneous, then all the remaining torsion coefficients should be constant. We calculate the structure equations for the system using (8.32) under the assumption that the S_* are constant.

$$\begin{aligned}
&\mathrm{d}^2\theta = 0, \quad \mathrm{d}^2\pi = 0, \\
&\mathrm{d}^2\omega = (-1 - S_2 - S_{\bar{2}}\overline{S_{\bar{2}}})\omega \wedge \pi \wedge \bar{\pi} + (-1 - S_2)\theta \wedge \omega \wedge \pi \\
&\qquad + (2S_1 + S_{\bar{2}}\overline{S_{\bar{2}}})\theta \wedge \bar{\pi} \wedge \pi + (2S_{\bar{2}} - S_{\bar{1}} + S_{\bar{2}}\overline{S_2})\pi \wedge \bar{\omega} \wedge \bar{\pi} \\
&\qquad + (3S_{\bar{1}} + S_1 + S_{\bar{2}}\overline{S_1})\bar{\theta} \wedge \bar{\pi} \wedge \pi + 2S_{\bar{1}}\theta \wedge \pi \wedge \bar{\theta} \\
&\qquad + 2S_{\bar{2}}\theta \wedge \pi \wedge \bar{\omega} + S_{\bar{2}}\pi \wedge \bar{\theta} \wedge \bar{\omega}.
\end{aligned} \tag{8.33}$$

From this it follows immediately that we should have $S_1 = S_{\bar{1}} = S_{\bar{2}} = 0$, $S_2 = -1$. There is only one homogeneous generic affine Darboux integrable system. The symmetry group is finite-dimensional and has dimension six. The structure equations are

$$\begin{aligned} \mathrm{d}\theta &= \pi \wedge \theta - \pi \wedge \omega + \bar{\pi} \wedge \bar{\theta}, \\ \mathrm{d}\omega &= (-\theta - \bar{\pi}) \wedge \omega - \pi \wedge \omega, \\ \mathrm{d}\pi &= \bar{\pi} \wedge \pi + (\theta \wedge \pi). \end{aligned} \tag{8.34}$$

8.4.4 Almost complex systems

The structure equations for the generic almost complex elliptic systems are given by

$$\begin{aligned} \mathrm{d}\theta &= -\alpha \wedge \theta - \pi \wedge \omega + \bar{\pi} \wedge \bar{\theta}, \\ \mathrm{d}\omega &= -(2\alpha - \bar{\alpha}) \wedge \omega - \pi \wedge \sigma, \\ \mathrm{d}\pi &= -(\bar{\alpha} - \alpha) \wedge \pi, \\ \mathrm{d}\alpha &= \pi \wedge \bar{\pi}. \end{aligned} \tag{8.35}$$

The remaining structure group is 2-dimensional and is given by matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a^2/\bar{a} & 0 \\ 0 & 0 & \bar{a}/a \end{pmatrix}$$

with a a non-zero complex number.

We continue by splitting the 1-form α into a real and imaginary part.

$$\alpha = \eta + i\zeta. \tag{8.36}$$

The new structure equations are

$$\begin{aligned} \mathrm{d}\theta &= -(\eta + i\zeta) \wedge \theta - \pi \wedge \omega + \bar{\pi} \wedge \bar{\theta}, \\ \mathrm{d}\omega &= -(\eta + 3i\zeta) \wedge \omega - \pi \wedge \sigma, \\ \mathrm{d}\pi &= 2i\zeta \wedge \pi, \\ \mathrm{d}\eta &= 0, \\ \mathrm{d}\zeta &= -i\pi \wedge \bar{\pi}, \end{aligned} \tag{8.37}$$

with

$$\sigma = S_1\theta + S_{\bar{1}}\bar{\theta} + S_2\omega + S_{\bar{2}}\bar{\omega}. \tag{8.38}$$

The action of the remaining structure group is by rotation of the complex numbers S_1, $S_{\bar{1}}$, S_2, $S_{\bar{2}}$. Let us first prove that S_2 cannot be zero on an open subset by calculating the structure equations for ω. We have

$$0 = \mathrm{d}^2\omega \equiv (\mathrm{d}S_2 + 2iS_2\zeta) \wedge \omega \wedge \pi - (3 + |S_{\bar{2}}|^2)\bar{\pi} \wedge \omega \wedge \pi \quad \mathrm{mod}\ \theta, \bar{\theta}, \bar{\omega}.$$

From this equation it follows that S_2 cannot be zero, since otherwise we would need $3 + |S_{\bar{2}}|^2 = 0$.

Ideal	Equation
$\theta, \bar{\theta}, \omega$	$8i\overline{Z_3}S_{\bar{2}} - \bar{S}_2 S_{\bar{2}} - S_{\bar{1}} = 0$
$\theta, \bar{\theta}, \bar{\omega}$	$\lvert S_{\bar{2}}\rvert^2 + 12\lvert Z_3\rvert^2 - 2i\overline{Z_3}S_2 = 0$
$\theta, \omega, \bar{\omega}$	$\overline{S_1}S_{\bar{2}} - 6i\overline{Z_3}S_{\bar{1}} - S_1 = 0$
$\bar{\theta}, \omega, \bar{\omega}$	$\overline{S_{\bar{1}}}S_{\bar{2}} - 4iS_1\overline{Z_3} = 0$

Table 8.1: Equations for the structure coefficients

We can arrange that S_2 is purely imaginary using the freedom in the structure group. The remaining structure group is only 1-dimensional and consists of matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with a real. The 1-form ζ reduces to a semi-basic form that we write as

$$\zeta = Z_1\theta + \overline{Z_1}\bar{\theta} + Z_2\omega + \overline{Z_2}\bar{\omega} + Z_3\pi + \overline{Z_3}\bar{\pi}.$$

The coefficients Z_j cannot be arbitrary functions since ζ still has to satisfy the structure equation $\mathrm{d}\zeta = -i\pi \wedge \bar{\pi}$.

On the reduced bundle the remaining 1-dimensional structure group leaves the structure coefficients S_* and Z_3 invariant. If our equation is homogeneous this implies these structure coefficients must be constant. We calculate $\mathrm{d}\zeta + i\pi \wedge \bar{\pi}$ modulo $\theta, \bar{\theta}, \omega, \bar{\omega}$. We find

$$\mathrm{d}\zeta + i\pi \wedge \bar{\pi} \equiv (i - 4i\lvert Z_3\rvert^2)\pi \wedge \bar{\pi} \equiv 0.$$

Hence Z_3 must have norm $1/2$.

By calculating $\mathrm{d}^2\omega$ modulo different ideals we find several equations for the structure coefficients S_* and Z_3. The generators of the ideals and the corresponding equations are given in Table 8.1. From the second equation it follows that Z_3 must be real and hence $Z_3 = \pm 1/2$. The choice for $Z_3 = 1/2$ or $Z_3 = -1/2$ corresponds to choosing one of the two (complex) characteristic systems. We continue our analysis with $Z_3 = 1/2$.

Remark 8.4.2. The branches with $Z_3 = -1/2$ are all equivalent under a discrete transformation of coframe to one of the equations with $Z_3 = 1/2$. The structure constants S_* are all transformed by complex conjugation. ⊘

Solving the system of equations for the S_* in Table 8.1, yields 4 classes of solutions. The four classes are labelled by suit, as was done in McKay [51, p. 89], and are given in Table 8.2.

In all different branches we have found values for Z_3 and the S_*. We still have to determine the values of Z_1 and Z_2. The fact that the remaining structure group still acts on these coefficients slightly complicates our analysis. By taking a suitable combination of the structure equations, we can find a coefficient that is invariant under the structure group. This leads to the conclusion that $Z_1 = 0$ in all cases. In the clubs case $Z_2 = r\exp(i\psi)$ with $\psi = \pi/4 - \phi/2$ and r a real function; in the other cases $Z_2 = 0$.

	$\clubsuit$	$\heartsuit$	$\spadesuit$	$\diamondsuit$
S_1	0	36	0	6
$S_{\bar{1}}$	0	$-24i$	0	$6i$
S_2	$-4i$	$-12i$	$-3i$	$-7i$
$S_{\bar{2}}$	$\exp(i\psi)$	3	0	-2

Table 8.2: Four cases for the homogeneous generic almost complex Darboux integrable equations

Clubs case. The structure group acts on r by a scaling, so r is not automatically constant. However we know that for any system of equations we need

$$\mathrm{d}^2\omega = 0, \quad \mathrm{d}\zeta + (i\pi \wedge \bar{\pi}) = 0.$$

We take a linear combination of these two expressions that does not involve η and $\mathrm{d}r$. A calculation using MAPLE yields

$$\mathrm{d}^2\omega + 3i(\mathrm{d}\zeta - (-i\pi \wedge \bar{\pi})) \wedge \omega = 16r(\exp(i(2\phi - \pi)/4)\omega \wedge \pi \wedge \bar{\omega} = 0.$$

This implies that $r = 0$, hence $Z_2 = 0$. With this the full structure equations become

$$\begin{aligned}
\mathrm{d}\theta &= -(\eta + i\zeta) \wedge \theta - \pi \wedge \omega + \bar{\pi} \wedge \bar{\theta}, \\
\mathrm{d}\omega &= -(\eta + 3i\zeta) \wedge \omega - \pi \wedge (-4i\omega + \exp(i\psi)\bar{\omega}), \\
\mathrm{d}\pi &= 2i\zeta \wedge \pi, \\
\mathrm{d}\eta &= 0,
\end{aligned} \tag{8.39}$$

with $\zeta = (1/2)(\pi + \bar{\pi})$. We have a continuous family of homogeneous Darboux integrable systems. Each class is labeled by $S_{\bar{2}} = \exp(i\psi)$ and has a 7-dimensional transitive symmetry group by Theorem 1.2.58.

Other cases. In the hearts, spades and diamonds case all structure equations are satisfied. We have a unique coframe on a 7-dimensional manifold with structure equations

$$\begin{aligned}
\mathrm{d}\theta &= -(\eta + i\zeta) \wedge \theta - \pi \wedge \omega + \bar{\pi} \wedge \bar{\theta}, \\
\mathrm{d}\omega &= -(\eta + 3i\zeta) \wedge \omega - \pi \wedge \sigma, \\
\mathrm{d}\pi &= 2i\zeta \wedge \pi, \\
\mathrm{d}\eta &= 0,
\end{aligned} \tag{8.40}$$

with $\zeta = (1/2)(\pi + \bar{\pi})$ and the coefficients in σ given in Table 8.2. Since all the structure functions are constant, the symmetry group is transitive and 7-dimensional.

	$\clubsuit$	$\heartsuit$	$\spadesuit$	$\diamondsuit$
S_1	0	36	0	6
$S_{\bar{1}}$	0	$-24i$	0	$6i$
S_2	$-4i$	$-12i$	$-3i$	$-7i$
$S_{\bar{2}}$	$\exp(i\psi)$	3	0	-2
Z_1	0	0	0	0
Z_2	0	0	0	0
Z_3	$1/2$	$1/2$	$1/2$	$1/2$

Table 8.3: Full table of coefficients the homogeneous generic almost complex Darboux integrable equations

Chapter 9

Pseudosymmetries

A symmetry is a transformation that preserves all structure present in an object. A pseudosymmetry is a transformation that preserves only part of the structure, but the part that is preserved *depends on the pseudosymmetry itself*. Let us try to explain this with a picture. Consider Figure 9.1. We have drawn schematically a vector field in the plane. The vector field has a symmetry in the horizontal direction. In the vertical direction there is no symmetry: if we translate the vectors in the vertical direction the vectors are not identical. However, if we translate the vectors in the vertical direction and then "forget about the projection direction", in other words project onto the horizontal line, then the vectors are the same. We say that the vertical translations define a pseudosymmetry for this vector field. This is different from saying that we are only looking at the horizontal component of the vector field. For example, consider the diagonal translations. These translations are not symmetries. The diagonal translations are also not pseudosymmetries, because the projection of the vectors onto the anti-diagonal is different. The diagonal translations are symmetries however for the horizontal component of the vector field.

In the remainder of this chapter we will give a more precise definitions of pseudosymmetries in the context of distributions and partial differential equations. But we think this picture of pseudosymmetries is worthwhile to keep in mind.

Our original interest for these pseudosymmetries comes from projections of second order partial differential equations to first order systems. These projections are a generalization of the symmetry methods developed by Lie to solve (partial) differential equations. See Hydon [42] for an introduction to these methods. The precise structure that a pseudosymmetry has to preserve depends on the application. In this chapter we discuss the pseudosymmetries of distributions (Section 9.1) and of (partial) differential equations (Section 9.2). After the definition of pseudosymmetries for partial differential equations we will describe a method to calculate pseudosymmetries for second order equations. In Section 9.5 we will discuss the relation of pseudosymmetries to integrable extensions and Bäcklund transformations.

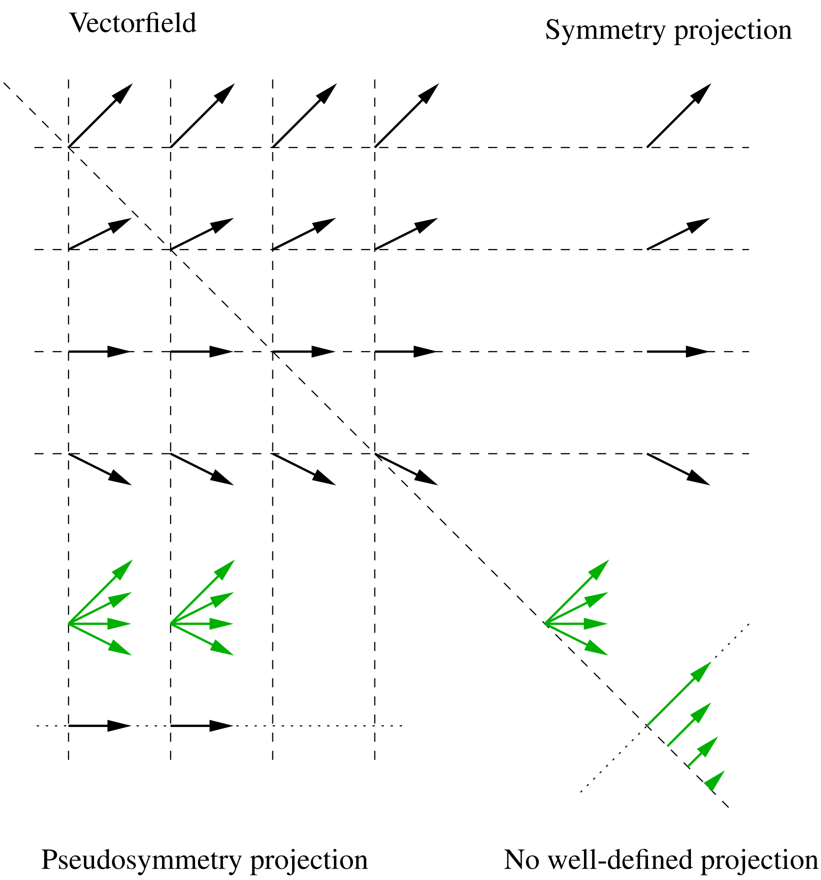

Figure 9.1: Vector field with pseudosymmetries

9.1 Pseudosymmetries of distributions

Let M be a manifold and $\mathcal{V} \subset TM$ a distribution. A (local) symmetry ϕ of $\mathcal{V}$ is a (local) diffeomorphism such that $\phi_*(\mathcal{V}) \subset \mathcal{V}$. Since ϕ is a diffeomorphism we have in fact $\phi_*(\mathcal{V}) = \mathcal{V}$. The infinitesimal version of a symmetry is a vector field V such that $[V, \mathcal{V}] \subset \mathcal{V}$.

For every non-zero vector field V we can locally form the quotient B of M by the integral curves of V. Let $\pi : M \to B$ be the projection. A symmetry V has the property that

$$\pi_*(\mathcal{V}_m) \subset T_{\pi(m)}B$$

depends only on the point $b = \pi(m) \in B$ and not on the point m in the fiber $\pi^{-1}(b)$. We say the distribution $\mathcal{V}$ *projects* down to a distribution $\mathcal{W} = \pi_*(\mathcal{V})$ on B. If V is transversal to $\mathcal{V}$, then $\mathcal{W}$ has the same rank as $\mathcal{V}$. If V is contained in $\mathcal{V}$, then the rank of $\mathcal{W}$ is the rank of $\mathcal{V}$ minus one.

This property of symmetries motivates the following definition.

Definition 9.1.1. Let M be a manifold and $\mathcal{V} \subset TM$ a distribution. A *pseudosymmetry* of $\mathcal{V}$ is a vector field V such that $[V, \mathcal{V}] \subset \mathcal{V} + V$. Here $\mathcal{V} + V$ indicates the distribution spanned by $\mathcal{V}$ and V. $\ominus$

If we take the quotient manifold B of M by the integral curves of a pseudosymmetry V, then locally we have a well-defined *projection* $\pi : M \to B$. If either $V \subset \mathcal{V}$ or V is pointwise not contained in $\mathcal{V}$, then the condition that V is a pseudosymmetry ensures that the bundle $\mathcal{V}$ projects to a constant rank distribution $\mathcal{W}$ on B.

To check that a vector field V is a (pseudo)symmetry of a distribution $\mathcal{V}$ we only need to check $[V, X] \equiv 0 \mod \mathcal{V}(+V)$ for a set of generators X of $\mathcal{V}$. The condition that $\mathcal{V}$ is a pseudosymmetry is not a linear condition. Hence the partial differential equations that express that $[V, X] \equiv 0 \mod \mathcal{V}+V$ are non-linear equations. This makes the computation of pseudosymmetries of a distribution more complicated than the calculation of the symmetries of a distribution.

For our applications we are also interested in more general objects. We are interested in integrable subbundles $\mathcal{U}$ such that

$$[\mathcal{U}, \mathcal{V}] \subset \mathcal{V} + \mathcal{U}. \tag{9.1}$$

If $\mathcal{U} \cap \mathcal{V}$ has constant rank, then locally the distribution $\mathcal{V}$ projects to a distribution on the quotient manifold $M/\mathcal{U}$. We call these bundles $\mathcal{U}$ *vector pseudosymmetries* of the distribution. More can be found in Section 9.4.

Lemma 9.1.2. *Let V be a symmetry or pseudosymmetry of the distribution $\mathcal{V}$. Then for any function f the vector field fV is a pseudosymmetry as well. Indeed, for $X \subset \mathcal{V}$*

$$[fV, X] = f[V, X] + X(f)V \subset \mathcal{V} + V.$$

The converse is not true; not every pseudosymmetry can be scaled to a symmetry. The space of symmetries is a vector space over the constants. The space of pseudosymmetries is in general not a vector space.

We have defined the concept of a pseudosymmetry as a vector field for which the integral curves locally define a projection. We want to stress that the important part of a pseudosymmetry is the *projection* that is (locally) given by taking the quotient of the manifold by the integral curves of the vector field. This is also reflected by the fact that multiplying a pseudosymmetry by a non-zero scalar function does not change the integral curves. Another definition of a pseudosymmetry of a distribution $\mathcal{V}$ would therefore be a rank one distribution $\mathcal{U}$ for which the condition (9.1) holds.

Example 9.1.3 (Pseudosymmetries of distributions).

- Let $M = \mathbb{R}^3$ with coordinates x, y, z and let $\mathcal{V}$ be the distribution spanned by the vector fields ∂_x and $\partial_y + z\partial_z$. Then $V_1 = \partial_x$ and $V_2 = \partial_y$ are a symmetries of $\mathcal{V}$ and $W = \partial_z$ is a pseudosymmetry of $\mathcal{V}$.

- Consider on $\mathbb{R}^4$ with coordinates x, y, z, u the following distributions:

$$\begin{aligned} \mathcal{V}_1 &= \operatorname{span}\big(\partial_x + y\partial_y\big), \quad \mathcal{V}_2 = \operatorname{span}\big(\partial_z + \partial_y\big), \\ \mathcal{V}_3 &= \operatorname{span}\big(\partial_x + \partial_u\big) \quad \text{and} \quad \mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3. \end{aligned}$$

  The vector field $V = \partial_y$ is a pseudosymmetry for $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}$. For each of the $\mathcal{V}_j$ we can find a suitable function ϕ_j such that $\phi_j\partial_y$ is a symmetry of $\mathcal{V}_j$. With a little work we can check that $e^{x-u}\partial_y$ is a true symmetry of all distributions.

- Let $M = \mathbb{R}^3$ and $\mathcal{V} = \operatorname{span}(\partial_x + y\partial_y + 2z\partial_z)$. Then both ∂_y and ∂_z are pseudosymmetries, but $\partial_y + \partial_z$ is not. On the other hand $\partial_y + y\partial_z$ is a pseudosymmetry. ⊘

Example 9.1.4 (True pseudosymmetry). On $\mathbb{R}^4$ with coordinates x, y, z, p define the distribution

$$\mathcal{V} = \operatorname{span}\big(\partial_y + \exp(z)\partial_z, \partial_x + p\partial_z, \partial_p\big)$$

and the vector field $V = \partial_z$. Then it is clear that V is a pseudosymmetry of the bundle $\mathcal{V}$. We try to find a function $\phi(x, y, z, p)$ such that ϕV is a symmetry. Then we must have

$$\begin{aligned} [\phi V, \partial_y + \exp(z)\partial_z] &= (\phi\exp(z) - \phi_y - \exp(z)\phi_z)\partial_z \equiv 0 \mod \mathcal{V}, \\ [\phi V, \partial_x + p\partial_z] &= -(\phi_x + p\phi_z)\partial_z \equiv 0 \mod \mathcal{V}, \\ [\phi V, \partial_p] &= -\phi_p\partial_z \equiv 0 \mod \mathcal{V}. \end{aligned}$$

From the last equation it follows $\phi_p = 0$. Then the second equation implies $\phi_z = 0$ and $\phi_x = 0$. The first equation reduces to $\phi\exp(z) - \phi_y = 0$. Since ϕ can only depend on y the only solution is $\phi = 0$. The pseudosymmetry is therefore a *true pseudosymmetry* in the sense that there is no symmetry having the same integral curves. ⊘

9.1.1 Invariant concepts

With a distribution we can associate many new objects in an invariant way. These objects, such as derived bundles and characteristic systems, are all invariant under symmetries of the distribution. We want to generalize as much of these concepts to pseudosymmetries as we can.

Theorem 9.1.5. *Let V be a pseudosymmetry of $\mathcal{V}$. Then V is a pseudosymmetry of the derived bundle $\mathcal{V}'$.*

Proof. Let $X \subset \mathcal{V}'$. We can assume that either $X \subset \mathcal{V}$ or $X = [Y, Z]$ with $Y, Z \subset \mathcal{V}$. In the first case it is clear that $[V, X] \subset \mathcal{V} + V \subset \mathcal{V}' + V$. In the second case we have $[V, [Y, Z]] = -[Y, [Z, V]] - [Z, [V, Y]] \subset [Y, \mathcal{V} + V] + [Z, \mathcal{V} + V] \subset \mathcal{V}' + V$. □

Unfortunately many other structures such as Cauchy characteristics and characteristic systems are not invariant under pseudosymmetries.

Example 9.1.6 (Pseudosymmetries and Cauchy characteristics). Let $M = \mathbb{R}^4$ with coordinates x, p, z, y . We define the distribution $\mathcal{V} = \operatorname{span}(\partial_x + p\partial_z, \partial_p, \partial_y)$. Then $C(\mathcal{V}) = \operatorname{span}(\partial_y)$. The vector field $X = \partial_z$ is a symmetry of $\mathcal{V}$. We have $[V, \mathcal{V}] \equiv 0 \mod \mathcal{V}$ and $[V, C(\mathcal{V})] \equiv 0 \mod C(\mathcal{V})$. Hence X is a symmetry of $C(\mathcal{V})$ as well. The vector field $V = \partial_z + y\partial_p$ is a pseudosymmetry of $\mathcal{V}$. But $[V, C(\mathcal{V})] = \operatorname{span}(\partial_p) \not\equiv 0 \mod C(\mathcal{V}) + V$, so V is not a pseudosymmetry of $C(\mathcal{V})$. This proves at the same time that V cannot be scaled to a true symmetry. ⊘

9.1.2 Formulation in differential forms

We can either work with distributions or differential forms. Since these objects are dual to each other every concept that can be (invariantly) defined in terms of distributions can be formulated in terms of differential forms and vice versa. The equivalent of the Lie bracket is the differential operator d; they are related by the formula $\mathrm{d}\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$. In the experience of the author pseudosymmetries are more easily described with vector fields, but for sake of completeness we will also give a description in terms of differential forms.

Let $\mathcal{V}$ be a vector subbundle of TM and let I be the ideal generated by the forms dual to $\mathcal{V}$, i.e., the 1-forms θ such that $\theta(X) = 0$ for all $X \in \mathcal{V}$. Generated here means generated as a $C^\infty(M)$-module in $\Omega^1(M)$.

A symmetry X of an ideal I is a vector field such that $\mathcal{L}_X(\theta) \equiv 0 \mod I$ for all $\theta \in I$. Define $\mathcal{W} = \mathcal{V} + X$ and let $J = \mathcal{W}^\perp$. Then X is a pseudosymmetry of $\mathcal{V}$ if and only if $\mathcal{L}_X\theta \equiv 0 \mod I$ for all $\theta \in J$. This is equivalent to $\mathcal{L}_X\theta \equiv 0 \mod I$ for all $\theta \in I$ with $\theta(X) = 0$.

9.1.3 Symmetries of pseudosymmetries

Let $\mathcal{V}$ be a distribution. For any pair of infinitesimal symmetries X, Y of $\mathcal{V}$ the Lie bracket $[X, Y]$ is again a symmetry. The infinitesimal symmetries of $\mathcal{V}$ form a Lie algebra S. Let V be a pseudosymmetry of $\mathcal{V}$. We can use the symmetries of $\mathcal{V}$ to generate new pseudosymmetries.

Theorem 9.1.7. *Let $\mathcal{V}$ be a distribution, V a pseudosymmetry and Φ a symmetry of the distribution. Then $\Phi_* V$ is a pseudosymmetry of $\mathcal{V}$.*

Proof. By definition of a symmetry we have $\Phi_* \mathcal{V} = \mathcal{V}$. Let $W = \Phi_*(V)$. We want to prove that $[W, Y] \subset \mathcal{V} + W$.

Since Φ is a diffeomorphism we have $[W, Y] = [\Phi_*(V), Y] = \Phi_*([V, \Phi_*^{-1}(Y)]) \subset \Phi_*([V, \mathcal{V}])$. But V is a pseudosymmetry of $\mathcal{V}$ and hence $\Phi_*([V, \mathcal{V}]) \subset \Phi_*(V + \mathcal{V}) = \Phi_*(V) + \mathcal{V}$. This shows that $[W, Y] \subset W + \mathcal{V}$. □

For any infinitesimal symmetry X of $\mathcal{V}$ the theorem above shows that $\exp(tX)V$ is a 1-parameter family of pseudosymmetries of $\mathcal{V}$. This suggests that by differentiation with respect to t at $t = 0$ we might find that $[X, V]$ is a pseudosymmetry for $\mathcal{V}$ as well. Example 9.1.8 below shows this is not true. For V, W pseudosymmetries the Lie brackets $[V, X]$ and $[V, W]$ are in general not pseudosymmetries.

Example 9.1.8. Consider the distribution on $\mathbb{R}^3$ with coordinates x, p, z generated by

$$\partial_x + p\partial_z, \quad \partial_p.$$

Then $V = x\partial_p + \partial_z$ is a pseudosymmetry and $X = \partial_x$ is a symmetry of the bundle. The commutator $[V, X] = -\partial_p$ is not a pseudosymmetry of the bundle. On the other hand, the flow of V by the vector field X is equal to $\exp(tX)_* V = (x + t)\partial_p + \partial_z$ and this is a pseudosymmetry for all t. ⊘

9.2 Pseudosymmetries for differential equations

In Chapter 4 we have seen that we can formulate the structure of first order systems and second order equations as distributions on the equation manifold. We will use this to define pseudosymmetries for certain classes of partial differential equations. These pseudosymmetries of partial differential equations all have in common that they are pseudosymmetries for the characteristic systems. This implies that if we locally take the quotient of the equation manifold by the pseudosymmetry, then the characteristic systems project to well-defined distributions on the quotient manifold.

9.2.1 Ordinary differential equations

Theorem 9.2.1. *Any pseudosymmetry of a rank one distribution that is not contained pointwise in the distribution is a multiple of a symmetry.*

Proof. Let the distribution be generated by a smooth non-zero vector field X. If V is a pseudosymmetry of the distribution, then $[V, X] \equiv gV \mod \mathcal{V}$ for a certain smooth function g. Then ϕV is a symmetry of the bundle if $[\phi V, X] \equiv \phi g V - X(\phi)V \equiv 0 \mod \mathcal{V}$. The equation $X(\phi) - g\phi = 0$ can be solved locally to give a non-zero solution ϕ. □

Example 9.2.2. The condition that V is pointwise not contained in $\mathcal{V}$ is necessary, since otherwise the function g might have singularities. Consider the distribution spanned by $X = \partial_x$ and the pseudosymmetry $V = x^2\partial_y + \partial_x$. The commutator is $[X, V] = 2x\partial_y \equiv 0 \mod \mathcal{V}, V$. For $x \neq 0$ we have $[X, V] \equiv 2x^{-1}V \mod \mathcal{V}$. We can scale the pseudosymmetry by the function x^{-2} at the points $x \neq 0$. The vector field $x^{-2}V = \partial_y + x^{-2}\partial_x$ is a symmetry of X, but not well-defined at $x = 0$. ⊘

Corollary 9.2.3. *Every pseudosymmetry of a scalar ordinary differential equation that is transversal to the flow of the equation is a multiple of a symmetry.*

Proof. Let x be the independent and z the dependent variable. Introduce coordinates x, z, $z_x = z_1$, $z_{xx} = z_2$, …, z_n for the equation manifold. Solving for the highest order variable we can write the equation as $z_n = F(x, z, z_j)$. The contact distribution is generated by the single vector field $\partial_x + z_1\partial_z + z_2\partial_{z_1} + \ldots z_{n-2}\partial_{z_{n-1}}$. The result follows from the previous theorem. □

The same result holds for certain classes of determined ordinary differential equations. So any interesting pseudosymmetries will have to be found for partial differential equations.

Example 9.2.4 (continuation of Example 9.1.4). Example 9.1.4 is in fact a geometric formulation of the following system M_1. Let z be a function of the variables x, y and define the equation $z_y = \exp(z)$. Let $J^1(\mathbb{R}^2)$ be the first jet bundle of $\mathbb{R}^2$. This bundle has coordinates $x, y, z, p = z_x, q = z_y$. Restriction of the contact ideal on $J^1(\mathbb{R}^2)$ to the equation manifold defined by $q = z_y = \exp(z)$ gives the system described in Example 9.1.4. Very closely related is the system M_2 defined by the ordinary differential equation $z' = e^z$, where z is a function of y.

For the first system M_1 the vector field ∂_z is a true pseudosymmetry (this was proven in the example above). For the second system M_2 we can multiply by e^z to obtain the symmetry $e^z\partial_z$. ⊘

9.2.2 Second order scalar partial differential equations

A second order scalar partial differential equation is defined in local coordinates by a smooth function F on the second order jet bundle $\mathrm{J}^2(\mathbb{R}^2)$. The equation manifold M is the 7-dimensional manifold defined by $F = 0$. The contact structure on $\mathrm{J}^2(\mathbb{R}^2)$ restricts to a contact structure on M and the dual system is a codimension 3 distribution $\mathcal{V} \subset TM$. Conversely, any codimension 3 distribution with some conditions defines a second order scalar partial differential equation. The conditions are given in Definition 4.1.1 and Theorem 4.1.2.

A symmetry for a second order scalar partial differential equation is a vector field V such that $[V, X] \subset \mathcal{V}$ for all $X \subset \mathcal{V}$. This implies that all structures derived from $\mathcal{V}$ are also preserved. For example the derived bundle $\mathcal{V}'$, the Cauchy-characteristics $C(\mathcal{V}')$, the splitting into characteristic subsystems $\mathcal{V}_+$ and $\mathcal{V}_-$, etc.

A pseudosymmetry is a vector field V on M that preserves the structure on M, modulo the vector field itself. If we take the quotient manifold $B = M/\operatorname{span}(V)$, then the structure on M should project to B. But what structures should be preserved? In any case the vector subbundle $\mathcal{V}$, since this is the defining structure for the partial differential equation. But preserving $\mathcal{V}$ might not be enough, since the fact that V is a pseudosymmetry for $\mathcal{V}$ does not automatically imply that V is a pseudosymmetry for all derived structures. We are interested in projecting and lifting solutions of partial differential equations. Motivated by this condition we say that

Definition 9.2.5. Let $(M, \mathcal{V})$ be a Vessiot system. A vector field V is a *pseudosymmetry* of the system if V is pointwise not contained in $\mathcal{V}$ and V is a pseudosymmetry of both Monge systems. $\ominus$

In the elliptic case we can also require that V preserves $\mathcal{V}$ and the complex structure on $\mathcal{V}$. The fact that V cannot lie in $\mathcal{V}$ itself implies that on the quotient system we again have two characteristic bundles of rank two.

Let V be a pseudosymmetry on $(M, \mathcal{V})$ and let B be the quotient system of M by the integral curves of V (this quotient might be defined only locally). The projection $M \to B$ is denoted by π. The characteristic systems $\mathcal{V}_\pm$ project to rank 2 characteristic systems $\mathcal{W}_\pm$ on B. The conditions on $\mathcal{V}_\pm$ imply that $[\mathcal{W}_+, \mathcal{W}_-] \equiv 0 \mod \mathcal{W}$ and both $\mathcal{W}_+$ and $\mathcal{W}_-$ are not integrable. The projected bundles $\mathcal{W}_\pm$ therefore define a hyperbolic first order system. Suppose that S is an integral surface for this first order system. The integral surface is foliated by the characteristic curves. The tangent spaces to the characteristic curves are given at each point by $TS \cap \mathcal{W}_\pm$. The inverse image $\pi^{-1}(S)$ has dimension three and $\mathcal{V}_S = T\pi^{-1}(S) \cap \mathcal{V} = (T\pi^{-1}(S) \cap \mathcal{V}_+) \oplus (T\pi^{-1}(S) \cap \mathcal{V}_-)$ is a rank 2 distribution on $\pi^{-1}(S)$. Since the distribution is integrable we can construct the integral surfaces using the Frobenius theorem. These integral surfaces have tangent space contained in $\mathcal{V}_S \subset \mathcal{V}$ and hence are integral surfaces for $\mathcal{V}$. In this way we have lifted an integral surface of the first order system to a family of integral surfaces of the Vessiot system.

Theorem 9.2.6. *Let V be a pseudosymmetry of a Vessiot system $(M, \mathcal{V})$. Then locally the quotient B of M by the integral curves of V is a first order system $(B, \mathcal{W})$. The projection $\pi : B \to M$ projects the Monge systems of $(M, \mathcal{V})$ to the Monge systems of $(B, \mathcal{W})$. There is a one-to-one correspondence between integral manifolds of $(B, \mathcal{W})$ and 1-dimensional families of integral manifolds of $(M, \mathcal{V})$.*

Example 9.2.7. Under the symmetry ∂_z the Laplace equation $z_{xx} + z_{yy}$ is projected to the Cauchy-Riemann equations $u_x = -v_y$, $u_x = v_x$. $\oslash$

For second order scalar equations in two independent variables we know that all internal symmetries are external symmetries. This is proved for example in Gardner and Kamran [38, Theorem 5.1]. Also all higher order contact symmetries are induced from first order symmetries. For pseudosymmetries the situation is a bit more complicated. Examples of true pseudosymmetries turned out to be rather difficult to find. For a pseudosymmetry there is no notion of prolongation, since a pseudosymmetry does not have to preserve the contact structure.

For second order scalar partial differential equations in two independent variables we have the following result.

Theorem 9.2.8. *Any pseudosymmetry of a hyperbolic second order scalar partial differential equation that is the prolongation of a first order contact transformation is a symmetry.*

Proof. The prolongation of any first order contact transformation is a second order contact transformation. If this is a pseudosymmetry, then the prolongation maps the equation manifold to itself. Then the pseudosymmetry is a contact transformation that preserves the equation manifold and hence a symmetry. □

This shows that any interesting pseudosymmetries should act non-trivially on the fibers of the projection to the first order contact manifold. Let us analyze the symmetries in more detail.

Theorem 9.2.9. *Any infinitesimal symmetry of a second order equation for which the corresponding first order contact transformation has no fixed points can be written as p_z for suitable coordinates x, y, z, p, q for the first order contact manifold.*

Proof. The equation manifold M has a natural projection $\pi : M \to P$ to a first order contact manifold. Any infinitesimal symmetry of the system is an infinitesimal contact transformation. This contact transformation must be a prolonged infinitesimal point transformation X of the first order contact manifold P. Since X has no fixed points we can write in suitable local coordinates as ∂_z, see Corollary 1.3.6. So we can choose coordinates x, y, z, p, q for P such that the contact structure on P is given by the contact form $\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y$ and X equals ∂_z. □

Remark 9.2.10. It is important that the infinitesimal symmetry has no fixed points on the first order contact manifold. For example consider the vector field $X = x^2\partial_z$ on Z with coordinates x, y, z. The prolongation to the first order jet bundle is $x^2\partial_z + 2x\partial_{z_x}$ and the prolongation to the second order jet bundle is $x^2\partial_z + 2x\partial_{z_x} + 2\partial_{z_{xx}}$. So the second prolongation of X has no fixed points, while X and the first prolongation of X have fixed points. ⊘

This restricts the class of first order systems that can be obtained by a symmetry reduction from a second order equation. Any symmetry can be locally written in the form $X = \partial_z$. The second order equation is then invariant under translations in the direction of z and can be written in the form $F(x, y, p, q, r, s, t) = 0$. The quotient of the equation by X is a first order system is of the form

$$q = r, \quad F(x, y, u, v, p, q, s) = 0,$$

for a function F. The first equation $q = r$ defines the Lagrangian subspace $\Lambda_b \subset \mathrm{Gr}_2(T_b B)$ at each point. Then the equation surface is a hypersurface in the Lagrangian subspace. These surfaces are far from generic. In particular the second order invariant I defined in (2.15) has value one or is undefined.

Example 9.2.11 (Scaling a pseudosymmetry to a symmetry). Consider the equation $r = t + p$. The distribution is $\mathcal{V} = \{ D_x, D_y, \partial_s, \partial_t \}$, with $D_x = \partial_x + p\partial_z + (t + p)\partial_p + s\partial_q$ and $D_y = \partial_y + q\partial_z + s\partial_p + t\partial_q$. The characteristic systems are given by

$$\mathcal{F} = \mathrm{span}\left(D_x + D_y + s\partial_s, \partial_s + \partial_t\right), \quad \mathcal{G} = \mathrm{span}\left(D_x - D_y + s\partial_s, \partial_s - \partial_t\right).$$

The vector field $V = \partial_z + \partial_p$ is a pseudosymmetry of the equation; this can be checked by a direct calculation. The vector field is not a prolonged first order contact transformation since V is not a symmetry of $\mathcal{V}$.

However when multiplied by a suitable function ϕ the vector field becomes a symmetry. We can also check this by calculation. Suppose that ϕV is a symmetry.

$$\begin{aligned}
[\phi V, D_x + D_y + s\partial_s] &\equiv \phi V - (D_x + D_y + s\partial_s)(\phi)V \mod \mathcal{F},\\
[\phi V, D_x - D_y + s\partial_s] &\equiv -(D_x - D_y + s\partial_s)(\phi)V \mod \mathcal{G},\\
[\phi V, \partial_s + \partial_t] &\equiv -\phi_s V \mod \mathcal{F},\\
[\phi V, \partial_s - \partial_t] &\equiv -\phi_t V \mod \mathcal{G}.
\end{aligned}$$

The last two equations imply $\phi_s = \phi_t = 0$ and then the first and second one together imply $\phi_q = \phi_p = \phi_z = \phi_y = 0$. From the first equation we see $\phi - \phi_x$, hence we can take $\phi = C\exp(x)$ as the general solution. The vector field $\exp(x)(\partial_z + \partial_p)$ is a symmetry of the system. ⊘

Remark 9.2.12. A method to find exact solutions of partial differential equations introduced (under a different name) by Bluman and Cole [10] is the use of *nonclassical symmetries*. For a more recent introduction see Hydon [42]. These nonclassical symmetries are defined on finite order jets bundles (in contrast with generalized symmetries, which are defined on infinite jet bundles), but are in general different from pseudosymmetries. The nonclassical symmetries also have the property that if X is a nonclassical symmetry, then λX is a nonclassical symmetry for every function λ. For second order scalar equations in two variables, the nonclassical symmetries are disjoint with the true pseudosymmetries. ⊘

Example 9.2.13 (True pseudosymmetry for second order equation). Consider the wave equation $s = 0$. The characteristic systems are given by

$$\mathcal{F} = \operatorname{span}\left(F_1 = \partial_x + p\partial_z + r\partial_p, F_2 = \partial_r\right), \quad \mathcal{G} = \operatorname{span}\left(G_1 = \partial_y + q\partial_z + t\partial_q, G_2 = \partial_t\right).$$

Define the vector field V by $\partial_z + p\partial_p + (p^2 + r)\partial_r$. We have

$$\begin{aligned}
[V, F_1] &\equiv pV \mod \mathcal{F},\\
[V, F_2] &= -\partial_r \equiv 0 \mod \mathcal{F},\\
[V, G_1] &= 0,\\
[V, F_2] &= 0.
\end{aligned}$$

Thus V is a pseudosymmetry. The vector field ϕV is a symmetry if and only if ϕ satisfies the system of equations $F_1\phi = p\phi$, $G_1\phi = 0$, $(p^2 + r)F_2(\phi) + \phi = 0$, $\phi_t = 0$. The only solution to this system is $\phi = 0$, hence V is a true pseudosymmetry. Note that V preserves the characteristic subsystem of $\mathcal{V}'$ and hence induces a first order vector field. This first order vector field $\partial_z + p\partial_p$ is not a contact transformation. ⊘

9.2.3 First order systems

For a first order system we define a pseudosymmetry as a vector field that is transversal to the contact distribution and is a pseudosymmetry of both characteristic Monge systems. The quotient of a first order system by a pseudosymmetry is a manifold P of dimension 5 and the Monge systems project to two rank two distributions $\mathcal{W}_\pm$. For elliptic equations the projected distributions $\mathcal{W}_\pm$ are distributions in the complexification of the tangent space. From the properties of the Monge systems it follows that $[\mathcal{W}_+, \mathcal{W}_-] \subset \mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_-$. For a hyperbolic first order system the projection is a hyperbolic exterior differential system of class $s = 1$. If the distribution $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_-$ is a contact distribution (maximally non-degenerate), then the projected system is a hyperbolic Monge-Ampère equation in the sense of Proposition 7.2.6.

Example 9.2.14 (Pseudosymmetry for the contact distribution). Consider the first order system $u_y = \exp(u), v_x = 0$. This system is contact equivalent to the first order wave equation (4.15), but in these coordinates the pseudosymmetries will be more simple. We introduce coordinates $x, y, u, v, p = u_x, s = v_y$ on the equation manifold. The system is hyperbolic and the two characteristic systems are given by

$$\mathcal{F} = \operatorname{span}\left(F_1 = \partial_x + p\partial_u, F_2 = \partial_p\right),$$
$$\mathcal{G} = \operatorname{span}\left(G_1 = \partial_y + \exp(u)\partial_u + s\partial_v + \exp(u)p\partial_p, G_2 = \partial_s\right).$$

Define the vector field $V = \partial_u$. Then we have

$$[V, F_1] = 0, \quad [V, F_2] = 0,$$
$$[V, G_1] \equiv \exp(u)V \mod \mathcal{V},$$
$$[V, G_2] = 0.$$

So V is a pseudosymmetry of $\mathcal{V}$. But $[V, G_1] = \exp(u)\partial_u + \exp(u)p\partial_p \not\equiv 0 \mod V, \mathcal{G}$. Thus V is not a pseudosymmetry of $\mathcal{G}$. This also proves that the pseudosymmetry V cannot be scaled to a symmetry of $\mathcal{V}$ by any factor. ⊘

Example 9.2.15 (continuation of Example 9.2.14). We continue with the first order system $u_y = \exp(u)$, $v_x = 0$. A pseudosymmetry of $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$ that is also a pseudosymmetry of the Monge systems is $W = \partial_u + p\partial_p$. In fact any vector field of the form $W + F(x, p\exp(-u), y + \exp(-u))\partial_p$ is a pseudosymmetry for $\mathcal{F}$ and $\mathcal{G}$. Note that W is a symmetry of the system when multiplied by $\exp(u)$. ⊘

Example 9.2.16. Take the first order system defined by $u_y = 0$, $v_x = u^2/2$. We write $p = u_x, s = v_y$. The characteristic systems are given by

$$\mathcal{F} = \operatorname{span}\big(\partial_x + p\partial_u + (u^2/2)\partial_v, \partial_p\big),$$
$$\mathcal{G} = \operatorname{span}\big(\partial_y + s\partial_v, \partial_s\big).$$

A true pseudosymmetry of the system is given by $V = \partial_u + \partial_v + u\partial_p$. Note that V is a symmetry of $\mathcal{G}$ and a pseudosymmetry of $\mathcal{F}$. Using the 1-parameter group of symmetries generated by the infinitesimal symmetry $x\partial_x + v\partial_v + p\partial_p + s\partial_q$ we find the family of pseudosymmetries $\partial_u + c\partial_v + u\partial_p$ for c an arbitrary constant. Another family of pseudosymmetries that can be obtained in this way, is given by $(1/4c^2u^2 - u + 1)\partial_u + \partial_v + (c^2pu/2 - c(u^2/2 - p) + u)\partial_p$. All these pseudosymmetries can be scaled by arbitrary functions.

Let $M/\operatorname{span}(V)$ be the quotient of M by the integral curves of the pseudosymmetry V. On the quotient manifold we can use coordinates $x, y, \tilde{u} = u - v, \tilde{p} = p - u^2/2, \tilde{s} = -s$. The projected characteristic systems are

$$\mathcal{F} = \operatorname{span}\left(\partial_x + \tilde{p}\partial_{\tilde{u}}, \partial_{\tilde{p}}\right), \quad \mathcal{G} = \operatorname{span}\left(\partial_y + \tilde{s}\partial_{\tilde{u}}, \partial_{\tilde{s}}\right).$$

This system is equivalent to the first order wave equation (Example 4.6.5). The general solution is given in parametric form by

$$\tilde{u} = \alpha(x) + \beta(y), \quad \tilde{p} = \alpha'(x), \quad \tilde{q} = \beta'(y).$$

This defines an integral manifold of the projected system that can be lifted to an integral manifold of the original system. If we make the coordinate transformation $\tilde{u} = u - v$, $\tilde{v} = v$, then the original system is transformed to the first order system $\tilde{u}_y + \tilde{v}_y = 0$, $\tilde{v}_x = (\tilde{u} + \tilde{v})^2/2$. Substitution of the general solution $\tilde{u} = \alpha + \beta$ into this system leads to an overdetermined first order system in $\tilde{v}$ for which the compatibility conditions are satisfied:

$$\tilde{v}_y = -\beta'(y), \quad \tilde{v}_x = \frac{(\alpha + \beta + \tilde{v})^2}{2}.$$

Integration of this system using the Frobenius theorem leads to the general solution for u and v. ⊘

The projected system for a pseudosymmetry of a first order system is a manifold of dimension 5 with two rank two distributions $\mathcal{W}_+, \mathcal{W}_-$. These bundles satisfy the conditions of a hyperbolic exterior differential system of class $s = 1$. We have the additional relations

- $[\mathcal{W}_+, \mathcal{W}_-] \subset \mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_-$,
- $\operatorname{rank}([\mathcal{W}, \mathcal{W}]) = 5$, hence either $\mathcal{W}_+$ or $\mathcal{W}_-$ is not integrable.

It could be that there are more relations that hold in general. The examples below show that there is quite some freedom in the projected structures.

Example 9.2.17 (Projected bundle). The total bundle $\mathcal{V} = \mathcal{W}_+ \oplus \mathcal{W}_-$ for the projected structure can be a contact distribution, but this is not always the case. Consider the system defined by $u_y = v_x = 0$, see Example 4.6.5. The vector field $\partial_u - \partial_v$ is a symmetry of this system. The quotient system has coordinates $x, y, p, s, z = u + v$ and the projected characteristic bundles are given by

$$\mathcal{W}_+ = \operatorname{span}\left(\partial_x + p\partial_z, \partial_p\right), \quad \mathcal{W}_- = \operatorname{span}\left(\partial_y + s\partial_z, \partial_s\right).$$

A form dual to the distribution $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_-$ is $\theta = \mathrm{d}z - p\mathrm{d}x - s\mathrm{d}y$. It is clear that this is a contact form.

If we take the symmetry ∂_v, then we find a projected system with coordinates x, y, u, p, s and characteristic systems $\mathcal{W}_+ = \operatorname{span}(\partial_x + p\partial_u, \partial_p)$, $\mathcal{W}_- = \operatorname{span}(\partial_y, \partial_s)$. The form dual to $\mathcal{W}_+ \oplus \mathcal{W}_-$ is equal to $\mathrm{d}u - p\mathrm{d}x$. ⊘

9.2.4 Monge-Ampère equations

Recall that in Section 7.2.1 we showed that a hyperbolic Monge-Ampère equation can be formulated as a rank 4 distribution $\mathcal{W}$ on a first order contact manifold P together with a hyperbolic structure on $\mathcal{W}$ that satisfied some additional properties.

Definition 9.2.18. A *pseudosymmetry* for a Monge-Ampère equation $(P, \mathcal{W}_+, \mathcal{W}_-)$ is a vector field V that is a pseudosymmetry of both characteristic systems. In formula:

$$[\mathcal{W}_\pm, V] \equiv 0 \mod \mathcal{W}_\pm, V. \tag{9.2}$$

⊖

If V is a pseudosymmetry for the Monge-Ampère equation $(P, \mathcal{W})$ pointwise not contained in $\mathcal{W}$, then locally we can define the quotient manifold $B = P/\operatorname{span}(V)$. The condition (9.2) implies that the bundles $\mathcal{W}_\pm$ project down to rank 2 bundles $\mathcal{U}_\pm$ on B. The projected bundles define an almost product structure or almost complex structure on B depending on the type of the equation. The integral manifolds of $(P, \mathcal{W})$ project to integral manifolds of this structure on B. Conversely, if we have an integral surface for the almost product or almost complex structure on B, then we can lift this to an integral surface of the Monge-Ampère equation. This lift depends only on the choice of a point in the fiber above a point on the surface.

Example 9.2.19 (Wave equation). On the first order jet bundle P we introduce coordinates x, y, z, p, q. The solutions of wave equation $z_{xy} = 0$ are in correspondence with the integral manifolds of the bundles $\mathcal{W}_\pm$ defined by

$$\mathcal{W}_+ = \operatorname{span}\left(\partial_x + p\partial_z, \partial_p\right), \quad \mathcal{W}_- = \operatorname{span}\left(\partial_y + q\partial_z, p_q\right).$$

The contact symmetries of this system depend on two functions of two variables. This will be proved in the examples in Section 9.3.2.

Consider the family of vector fields depending on two functions α, β defined by

$$V = \partial_z + \beta(x, p)(\partial_x + p\partial_z) - \alpha\partial_p. \tag{9.3}$$

We scale V by a function ϕ. A simple calculation yields that

$$\begin{aligned} [\partial_x + p\partial_z, \phi V] &\equiv (\phi_x + p\phi_z)V + \phi\alpha V \mod \mathcal{W}_+, \\ [\partial_p, \phi V] &\equiv \phi_p V + \phi\beta V \mod \mathcal{W}_+, \\ [\partial_y + q\partial_z, \phi V] &\equiv (\phi_y + q\phi_z)V \mod \mathcal{W}_-, \\ [\partial_q, \phi V] &\equiv \phi_q V \mod \mathcal{W}_-. \end{aligned} \tag{9.4}$$

Hence V is a pseudosymmetry for all functions α, β. The condition that V can be scaled to a true symmetry is given by the system of equations

$$\begin{gathered} (\phi_x + p\phi_z) + \phi\alpha = 0, \quad \phi_p V + \phi\beta = 0, \\ \phi_y + q\phi_z = 0, \quad \phi_q = 0. \end{gathered}$$

From this system it immediately follows that ϕ is a function of the variables x, p and should satisfy

$$\phi_x = -\phi\alpha, \quad \phi_p = -\phi\beta. \tag{9.5}$$

The compatibility conditions for this system are $\alpha_p = \beta_x$, see Example 1.2.20. ⊘

9.3 Decomposition method

We want to calculate the pseudosymmetries of a second order scalar partial differential equation in two independent variables. The quotient of the equation by this pseudosymmetry will be a first order system. First we give a geometric formulation of both systems and then we give a method for finding pseudosymmetries. The idea for the method was developed by Robert Bryant and the author during a visit of the author to Robert Bryant at Duke University in July 2005. We also give several examples. We develop the theory for hyperbolic systems. A similar construction for elliptic systems is possible, with the complication that one has to work with complex systems.

9.3.1 Solution method

Suppose we have a second order partial differential equation and want to find all pseudosymmetries. Writing down a general vector field (modulo a scalar factor) and then writing down the conditions for this vector field to be a pseudosymmetry gives a system of 16 equations for 6 unknown functions of 7 variables. Solving this system is quite difficult in practice.

Let $(M, \mathcal{V})$ be hyperbolic system determined by a second order scalar partial differential equation in two independent variables. Then we can choose an adapted coframing $\theta^0, \theta^1, \theta^2, \omega^3, \omega^4, \omega^5, \omega^6$ such that $\mathcal{V}$ is the dual distribution to $\theta^0, \theta^1, \theta^2$ and

$$\mathrm{d}\theta^0 \equiv \theta^1 \wedge \omega^3 + \theta^2 \wedge \omega^4 \mod \theta^0, \tag{9.6a}$$
$$\mathrm{d}\theta^1 \equiv \omega^3 \wedge \omega^5 \mod \theta^0, \theta^1, \theta^2, \tag{9.6b}$$
$$\mathrm{d}\theta^2 \equiv \omega^4 \wedge \omega^6 \mod \theta^0, \theta^1, \theta^2. \tag{9.6c}$$

See Definition 6.1.4. We will write $\mathcal{F} = \mathcal{V}_+ = \operatorname{span}(\partial_{\omega^3}, \partial_{\omega^5})$ and $\mathcal{G} = \mathcal{V}_- = \operatorname{span}(\partial_{\omega^4}, \partial_{\omega^6})$ for the Monge systems.

Lemma 9.3.1. *Any pseudosymmetry (and hence every symmetry) of a hyperbolic, elliptic or parabolic second order equation is pointwise not contained in the derived bundle of the contact distribution on an open dense subset.*

Proof. Let $(M, \mathcal{V})$ be the equation manifold with contact distribution $\mathcal{V}$. Let $\mathcal{V}'$ be the derived bundle and $\mathcal{V}''$ the second derived bundle. We have $\dim \mathcal{V}'' = 7$, so $\mathcal{V}'' = TM$. Let V be a pseudosymmetry of $\mathcal{V}$. Then the non-degeneracy of $\mathcal{V}$ implies that V is pointwise not contained in $\mathcal{V}$ on an open dense subset. The vector field V must also be a pseudosymmetry of $\mathcal{V}'$, see Theorem 9.1.5. Assume V is contained in $\mathcal{V}'$ on a non-empty open subset of M.

Then since V is a pseudosymmetry of $\mathcal{V}'$ we have $[V, X] \subset \mathcal{V}'$ for all $X \subset \mathcal{V}'$. But then V is in $C(\mathcal{V}')$. For these systems $C(\mathcal{V}') \subset \mathcal{V}$ and this gives a contradiction. □

Remark 9.3.2. For a second order equation and a pseudosymmetry V there can be closed sets for which $V_x \in \mathcal{V}_x$. For example take the wave equation described in Example 4.2.2. The vector field $X = \partial_x$ is the infinitesimal symmetry of the wave equation that generates the translations in the x-direction. The characteristic bundle $\mathcal{F}$ is spanned by $\partial_x + p\partial_z + r\partial_p, \partial_r$. For all points m for which $p = r = 0$ we have $(\partial_x)_m \in \mathcal{F}_m$. ⊘

From here on we will concentrate on pseudosymmetries V for which $V_m \notin \mathcal{V}_m$ for all points m. These pseudosymmetries are interesting because they generate nice projections to first order systems. The lemma above shows that for any pseudosymmetry the set of points where this condition might not be satisfied is a closed set of lower dimension.

Assuming that $V_m \notin \mathcal{V}_m$ we can adapt the coframing such that $\theta^0(V) = 1$ and $\theta^{1,2}(V) = \omega^{3,4,5,6}(V) = 0$. Any vector field on TM determines a section of the bundle $TM/\mathcal{V}$. We define the map

$$\rho : \mathscr{X}(M) \to \Gamma(TM/\mathcal{V}) : X \mapsto \tilde{X} = X \mod \mathcal{V}.$$

Lemma 9.3.3. *Any pseudosymmetry V is uniquely determined by the section $\tilde{V} = \rho(V)$ of $TM/\mathcal{V}$.*

Proof. Suppose V and $W = V + X$ are both pseudosymmetries of $(M, \mathcal{V})$ and $\tilde{W} = \tilde{V}$. Then $X \subset \mathcal{V}$. Since V and W are pseudosymmetries we have $[V, \mathcal{V}] \subset \mathcal{V} + \mathbb{R}V$ and $[W, \mathcal{V}] \subset \mathcal{V} + \mathbb{R}V$. But then also $[X, \mathcal{V}] \subset \mathcal{V} + \mathbb{R}V$. Since X is in $\mathcal{V}$ and $V \not\subset \mathcal{V}'$ this implies that $[X, \mathcal{V}] \subset \mathcal{V}$. But the bundle $\mathcal{V}$ has no Cauchy characteristics so this implies $X = 0$ and hence $V = W$. □

So a pseudosymmetry is uniquely determined by the corresponding section in $TM/\mathcal{V}$. In turn such a section uniquely determines a basis for $TM/\mathcal{V}$ up to scalar factors. The bundles $\mathcal{V}'_+/\mathcal{V}$ and $\mathcal{V}'_-/\mathcal{V}$ are invariant bundles in $TM/\mathcal{V}$ and $TM/\mathcal{V} = \operatorname{span}(\mathcal{V}) \oplus (\mathcal{V}'_+/\mathcal{V}) \oplus (\mathcal{V}'_-/\mathcal{V})$. By choosing non-zero vectors V_1 in $(\mathcal{V}'_+/\mathcal{V})$ and V_2 in $(\mathcal{V}'_-/\mathcal{V})$ we have a basis V, V_1, V_2 for $TM/\mathcal{V}$ that is invariant up to scalar factors. We already have invariantly defined the system $M^0 = \operatorname{span}(\theta^0)$ dual to $\mathcal{V}'$ and the two systems $M^1 = \operatorname{span}(\theta^0, \theta^1)$ and $M^2 = \operatorname{span}(\theta^0, \theta^2)$. We choose new differential forms θ^1 and θ^2 in M^1 and M^2, respectively, such that

$$\theta^1(V) = 0, \quad \theta^2(V) = 0. \tag{9.7}$$

Given the section $\tilde{V}$ of $TM/\mathcal{V}$ this determines θ^0, θ^1 and θ^2 up to scalar factors. The fact that $\tilde{V}$ determines θ^1 and θ^2 uniquely, suggest looking for a condition on θ^1 and θ^2 such that $\tilde{V}$ is a pseudosymmetry.

Lemma 9.3.4. *Let V be a pseudosymmetry of $(M, \mathcal{V})$ and define θ^1, θ^2 as in* (9.7). *Then the equations*

$$(\mathrm{d}\theta^1)^2 \wedge \theta^1 \wedge \theta^2 = 0, \quad (\mathrm{d}\theta^2)^2 \wedge \theta^1 \wedge \theta^2 = 0 \tag{9.8}$$

hold.

Proof. The equation $(d\theta^1)^2 \wedge \theta^1 \wedge \theta^2 = 0$ can be rephrased as $d\theta^1$ decomposes modulo θ^1, θ^2 or $d\theta^1$ is a decomposable 2-form on $\mathcal{V} = \operatorname{span}(\theta^1, \theta^2)^\perp$. If $d\theta^1$ does not decompose modulo θ^1, θ^2, then we have

$$d\theta^1 \equiv \omega^3 \wedge \omega^5 + \alpha \wedge \theta^0 \quad \bmod \theta^1, \theta^2$$

for some 1-form α that is non-zero modulo $\theta^0, \theta^1, \theta^2, \omega^3, \omega^5$. So α is non-zero when restricted to the distribution $\mathcal{V}_- = \operatorname{span}(\partial_{\omega^4}, \partial_{\omega^6})$. In particular we can assume that $\alpha = c_4\omega^4 + c_6\omega^6$ and not all c_j are zero. Assume (without loss of generality) that $c_4 \neq 0$. Then we have

$$\theta^1([V, \partial_{\omega^4}]) = -d\theta^1(V, \partial_{\omega^4}) = c_4 \neq 0.$$

So $[V, \mathcal{V}_-] \not\subset \mathcal{V}_- + \mathbb{R}V$ and hence V is not a pseudosymmetry of the characteristic bundle $\mathcal{V}_-$. This proves that $(d\theta^1)^2 \wedge \theta^1 \wedge \theta^2 = 0$. A similar argument shows $(d\theta^2)^2 \wedge \theta^1 \wedge \theta^2 = 0$.□

So if V is a pseudosymmetry, then there are $d\theta^1$ and $d\theta^2$ that decompose modulo θ^1, θ^2. Recall that the structure equations (9.6) only require θ^1 and θ^2 to decompose modulo $\theta^0, \theta^1, \theta^2$.

Remark 9.3.5. A more geometric proof of the lemma above is as follows. If V is a pseudosymmetry of a second order equation, then the quotient system will be a first order system. For this first order system we can find an adapted coframing with characteristic forms $\tilde{\theta}^1, \tilde{\theta}^2$. Recall that for first order systems the characteristic contact forms satisfy the decomposition equation

$$(d\tilde{\theta}^1)^2 \wedge \tilde{\theta}^1 \wedge \tilde{\theta}^2.$$

The pullbacks $\pi^*\tilde{\theta}^1$, $\pi^*\tilde{\theta}^2$ of $\tilde{\theta^1}$ and $\tilde{\theta^2}$ are equal to θ^1 and θ^1 modulo a term θ^0. But since $\theta^1(V) = 0$ and $\pi^*\tilde{\theta}^1(V) = 0$ (since the pullbacks are semi-basic) they must be the same. The decomposition equation for $\tilde{\theta}^1, \tilde{\theta}^2$ can also be pulled back and this provides the decomposition equation for θ^1, θ^2. ⊘

Now we come the the key idea of the method. In order for the vector field V dual to θ^0 to be a pseudosymmetry it is necessary and sufficient that $d\theta^1$ and $d\theta^2$ decompose modulo θ^1, θ^2.

Theorem 9.3.6. *Let θ^0, θ^1, θ^2, ω^3, ω^4, ω^5, ω^6 be an adapted coframing* (9.6)*. The vector field $V = \partial_{\theta^0}$ dual to θ^0 is a pseudosymmetry for the equation if and only if*

$$(d\theta^1)^2 \wedge \theta^1 \wedge \theta^2 = 0, \quad (d\theta^2)^2 \wedge \theta^1 \wedge \theta^2 = 0. \tag{9.9}$$

Proof. The necessity of the conditions (9.9) was already proved in Lemma 9.3.4. We will prove that the conditions are sufficient. The characteristic subsystems are spanned by $\mathcal{F} = \{\partial_{\omega^3}, \partial_{\omega^5}\}$ and $\mathcal{G} = \{\partial_{\omega^4}, \partial_{\omega^6}\}$. We will prove that $[V, \mathcal{F}] \equiv 0 \mod V, \mathcal{F}$. The case

$[V, \mathcal{G}] \equiv 0 \mod V, \mathcal{G}$ can be proved in a similar way. The condition we have to prove is equivalent to

$$\begin{aligned}
\theta^1([V, \partial_{\omega^3}]) &= 0, & \theta^1([V, \partial_{\omega^5}]) &= 0,\\
\theta^2([V, \partial_{\omega^3}]) &= 0, & \theta^2([V, \partial_{\omega^5}]) &= 0,\\
\omega^4([V, \partial_{\omega^3}]) &= 0, & \omega^4([V, \partial_{\omega^5}]) &= 0,\\
\omega^6([V, \partial_{\omega^3}]) &= 0, & \omega^6([V, \partial_{\omega^5}]) &= 0.
\end{aligned} \tag{9.10}$$

From these eight equations we will prove that 2 representative ones are satisfied, the other six are similar. Using equation (A.4) we have

$$\begin{aligned}
\theta^1([V, \partial_{\omega^3}]) &= -\mathrm{d}\theta^1(V, \partial_{\omega^3}) + V(\theta^1(\partial_{\omega^3})) - \partial_{\omega^3}(\theta^1(V))\\
&= -\mathrm{d}\theta^1(V, \partial_{\omega^3}) + V(0) - \partial_{\omega^3}(0)\\
&= (\omega^5 \wedge \omega^3)(V, \partial_{\omega^3}) = 0.
\end{aligned}$$

So the first equation is satisfied. Next we consider the fifth equation.

$$\begin{aligned}
\omega^4([V, \partial_{\omega^3}]) &= -\mathrm{d}\omega^4(V, \partial_{\omega^3}) + V(\omega^4(\partial_{\omega^3})) - \partial_{\omega^3}(\omega^4(V))\\
&= -\mathrm{d}\omega^4(V, \partial_{\omega^3}) + V(0) - \partial_{\omega^3}(0)\\
&= -\mathrm{d}\omega^4(V, \partial_{\omega^3}).
\end{aligned}$$

From the structure equations we find

$$0 = \mathrm{d}^2\theta^2 \equiv \mathrm{d}\omega^4 \wedge \omega^6 - \omega^4 \wedge \mathrm{d}\omega^6 \mod \theta^1, \theta^2, \omega^3 \wedge \omega^5, \omega^4 \wedge \omega^6.$$

This implies that $\mathrm{d}\omega^4 \equiv 0 \mod \theta^1, \theta^2, \omega^4, \omega^6, \omega^5$ and therefore $\omega^4([V, \partial_{\omega^3}]) = 0$. The other six equations from (9.10) are similar. □

Finding the pseudosymmetries is therefore equivalent to finding the coframings that are adapted properly in the sense that $\mathrm{d}\theta^1$ and $\mathrm{d}\theta^2$ decompose modulo θ^1, θ^2. This condition is equivalent to

$$\begin{aligned}
(\mathrm{d}\theta^1)^2 \wedge \theta^1 \wedge \theta^2 &= 0,\\
(\mathrm{d}\theta^2)^2 \wedge \theta^1 \wedge \theta^2 &= 0.
\end{aligned} \tag{9.11}$$

The pair θ^1, θ^2 then determines a section $\tilde{V}$ of $TM/\mathcal{V}$ up to a scalar factor and by Lemma 9.3.3 there is a unique pseudosymmetry V.

We can parameterize all possible choices of adapted θ^1, θ^2 using two functions p_1, p_2 and the transformation $\theta^1 \mapsto \theta^1 + p_1\theta^0$, $\theta^2 \mapsto \theta^2 + p_2\theta^0$. The condition that the 2-forms $\mathrm{d}\theta^1$ and $\mathrm{d}\theta^2$ decompose translates to a system of partial differential equations for p_1 and p_2.

The good thing about this method for finding pseudosymmetries is that the calculation has been split into two steps. The first step is to determine a pair of adapted contact forms θ^1, θ^2 that satisfies the decomposition equation (9.11). This corresponds to choosing a suitable section $\tilde{V}$ of $TM/\mathcal{V}$. To find such a pair we have to solve a system of equations for two unknown functions p_1, p_2. The second step is to extend $\tilde{V}$ to a pseudosymmetry V. The theory above proves this extension always exists and is unique. In practice we search for an adapted coframing, i.e., we have to adapt ω^3, ω^4, ω^5, ω^6 by calculating $\mathrm{d}\theta^1$ and $\mathrm{d}\theta^2$. Then we can define V as the dual vector field to θ^0.

Other advantages are that the system of partial differential equations for p_1, p_2 is a system of only 4 equations for two unknown functions, instead of 16 equations for 7 functions if we start with a general vector field. Also we have formulated our problems in terms of differential forms and can therefore use Cartan's method of equivalence and the Cartan-Kähler theorem to tackle the problem. An example of the method is given in the examples 9.3.8 and 9.3.13.

In general the equations (9.11) yield four quasi-linear equations for the 2 unknown functions p_1, p_2. This system is overdetermined, so in general there will be no pseudosymmetries. We work out these equations explicitly. Assume that we have an adapted coframing as in (9.6). We write $\omega^0 = \theta^0$, $\omega^{1,2} = \theta^{1,2} + p_{1,2}\theta^0$ and $\Omega = \omega^0 \wedge \omega^1 \wedge \omega^2$. There exist 1-forms a_j, b_j, c_j, $j = 0, 1, 2$ with $b_0 = \omega^3$, $c_0 = \omega^4$ such that the structure equations for θ^j are given by

$$\begin{aligned}
\mathrm{d}\theta^0 &= a_0\theta^0 + b_0\theta^1 + c_0\theta^2, \\
\mathrm{d}\theta^1 &= \omega^3 \wedge \omega^5 + a_1 \wedge \theta^0 + b_1 \wedge \theta^1 + c_1 \wedge \theta^2, \\
\mathrm{d}\theta^2 &= \omega^4 \wedge \omega^6 + a_2 \wedge \theta^0 + b_2 \wedge \theta^1 + c_2 \wedge \theta^2.
\end{aligned} \tag{9.12}$$

Let $\Theta = \theta^0 \wedge \theta^1 \wedge \theta^2$. The first decomposition equation can be written as

$$\begin{aligned}
0 &= (\mathrm{d}\omega^1)^2 \wedge \omega^1 \wedge \omega^2 \\
&= \big(\mathrm{d}\theta^1 + \mathrm{d}p_1 \wedge \theta_0 + p_1\mathrm{d}\theta_0\big)^2 \wedge (\theta^1 + p_1\theta^0) \wedge (\theta^2 + p_2\theta^0) \\
&= 2\omega^3 \wedge \omega^5 \wedge \mathrm{d}p_1 \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \\
&\quad + \big(\omega^3 \wedge \omega^5 + a_1 \wedge \theta^0 + b_1 \wedge \theta^1 + c_1 \wedge \theta^2 + p_1(a_0\theta^0 + b_0\theta^1 + c_0\theta^2)\big)^2 \\
&\qquad \wedge (\theta^1 + p_1\theta^0) \wedge (\theta^2 + p_2\theta^0) \\
&= 2\mathrm{d}p_1 \wedge \Theta \wedge \omega^3 \wedge \omega^5 + 2(a_1 + p_1a_0) \wedge \Theta \wedge \omega^3 \wedge \omega^5 \\
&\quad - 2p_1(b_1 + p_1b_0) \wedge \Theta \wedge \omega^3 \wedge \omega^5 - 2p_2(c_1 + p_1c_0) \wedge \Theta \wedge \omega^3 \wedge \omega^5 \\
&= 2\big(\mathrm{d}p_1 + a_1 + p_1(a_0 - b_1) - p_1^2b_0 - p_2c_1 - p_1p_2c_0\big) \wedge \Theta \wedge \omega^3 \wedge \omega^5.
\end{aligned}$$

And the second decomposition equation reduces to

$$\begin{aligned}
0 &= \big(\mathrm{d}\theta^2 + \mathrm{d}p_2 \wedge \theta^0 + p_2\mathrm{d}\theta_0\big)^2 \wedge (\theta^1 + p_1\theta^0) \wedge (\theta^2 + p_2\theta^0) \\
&= 2\big(\mathrm{d}p_2 + a_2 + p_2(a_0 - c_2) - p_2^2b_0 - p_1c_2 - p_1p_2b_0\big) \wedge \Theta \wedge \omega^4 \wedge \omega^6.
\end{aligned}$$

In local coordinates there are 4 first order equations for the two unknown functions p_1 and p_2. The equations are quasi-linear and polynomial in p_1, p_2.

Remark 9.3.7 (Cartan's test). If we add the functions p_1, p_2 as new variables to the system, then we are looking for 7-dimensional integral planes of the exterior differential ideal $\mathcal{I}$ generated by the two 6-forms (9.11) defined above. The space of integral elements $V_7(\mathcal{I})$ has codimension 4, since we have 4 first order equations in p_1, p_2.

Let $\pi^1 = \mathrm{d}p_1, \pi^2 = \mathrm{d}p_2$. We can define the integral flag

$$
\begin{aligned}
E_1 = \{\partial_{\omega^0}\} \subset E_2 = \{\, \partial_{\omega^0}, \partial_{\omega^1} \,\} \subset \ldots, \\
\ldots \subset E_7 = \{\partial_{\omega^0}, \partial_{\omega^1}, \partial_{\omega^2}, \partial_{\omega^3} + p_1 p_2 \partial_{\pi^2}, \partial_{\omega^5}, \partial_{\omega^4} + p_1 p_2 \partial_{\pi^1}, \partial_{\omega^6}\}.
\end{aligned}
$$

The codimensions of the polar spaces are $c_0 = c_1 = c_2 = c_3 = c_4 = 0$, $c_5 = 1$, $c_6 = 1$. Cartan's test is not satisfied, hence the system is not in involution. ⊘

Example 9.3.8 (Liouville equation). We will calculate the pseudosymmetries of the Liouville equation $s = \exp(z)$. We start with the initial coframing

$$
\begin{aligned}
\omega^0 &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\
\omega^1 &= \mathrm{d}p - r\mathrm{d}x - e^z\mathrm{d}y, \\
\omega^2 &= \mathrm{d}q - e^z\mathrm{d}x - t\mathrm{d}y, \\
\omega^3 &= \mathrm{d}x, \quad \omega^4 = \mathrm{d}y, \\
\omega^5 &= \mathrm{d}r - e^z p\mathrm{d}y, \quad \omega^6 = \mathrm{d}t - e^z q\mathrm{d}x.
\end{aligned}
$$

In the notation of (9.12) above we have $a_0 = 0$, $b_0 = \omega^3$, $c_0 = \omega^4$, $a_1 = e^z\omega^4$, $b_1 = c_1 = 0$, $a_2 = e^z\omega^3$, $b_2 = c_2 = 0$. The conditions for p_1 and p_2 are

$$
\begin{aligned}
\mathrm{d}p_1 + e_z\omega^4 - p_1 p_2 \omega^4 &\equiv 0 \mod \omega^{0,1,2,3,5}, \\
\mathrm{d}p_2 + e_z\omega^3 - p_1 p_2 \omega^3 &\equiv 0 \mod \omega^{0,1,2,4,6}.
\end{aligned}
$$

A special family of solutions is given by $p_2 = qh(y) - h(y)^2 + h'(y) - t)/(q - h(y))$, $p_1 = e^z \partial p_2 / \partial t$ for an arbitrary function $h(y)$. There is a similar family of solutions with p_1 and p_2 interchanged. The family of pseudosymmetries determined by the family is

$$
\begin{aligned}
\partial_y + &h(y)\partial_z + \Big(qh(y) - h(y)^2 + h'(y)\Big)\,\partial_q \\
&+ \Big(h(y)^3 - qh(y)^2 + 2th(y) - 3h(y)h'(y) + qh'(y) + h''(y)\Big)\,\partial_t.
\end{aligned}
$$

We can scale these by a factor $\exp(-\int^y h(y)\mathrm{d}y)$ to be true symmetries of the system. ⊘

Remark 9.3.9. There are no apparent generalizations of the method described in this section to other types of partial differential equations. The method for finding pseudosymmetries is closely related to the structures of second order scalar equations and first order systems, which are very similar. Also the contact forms $\theta^0, \theta^1, \theta^2$ are special in the sense that they together already determine the structure of the system. For pseudosymmetries of first order systems the resulting systems are not determined by the projected distribution alone. The decomposition into characteristic systems has to be given. ⊘

9.3.2 Pseudosymmetries of the wave equation

Example 9.3.10 (Contact symmetry group of the wave equation). The wave equation is a Monge-Ampère equation that can be formulated as a hyperbolic exterior differential system $\mathcal{I}$ of class $s = 1$ on the first order contact manifold with variables x, y, z, p, q. The system $\mathcal{I}$ is generated by

$$\theta = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \quad \Psi^1 = \mathrm{d}x \wedge \mathrm{d}p, \quad \Psi^2 = \mathrm{d}y \wedge \mathrm{d}q.$$

See for example Ivey and Landsberg [43, Section 6.4]. The forms θ, Ψ^1, Ψ^2 are unique up to a scalar factor. We want to show that the group of contact symmetries of this system depends on two functions of two variables.

Let σ be a contact symmetry. For simplicity we assume that we have labelled the two characteristic systems and σ does not interchange the two characteristic systems. Then we must have

$$\sigma^*(\mathrm{d}x \wedge \mathrm{d}p) = C(x, p)\mathrm{d}x \wedge \mathrm{d}p, \quad \sigma^*(\mathrm{d}y \wedge \mathrm{d}q) = D(y, q)\mathrm{d}y \wedge \mathrm{d}q,$$

for functions C, D and hence any σ must be of the form $\tilde{x} = a(x, p)$, $\tilde{p} = b(x, p)$, $\tilde{y} = m(y, q)$ and $\tilde{q} = n(y, q)$. The contact form θ must be preserved, hence we have

$$\sigma^*(\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y) = \mu(\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y),$$

for a function μ. By taking the exterior derivative we find

$$\sigma^*(\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q) = \mathrm{d}\mu \wedge \theta + \mu(\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q).$$

From this is follows that $\mathrm{d}\mu \equiv 0 \mod \theta$. Since $\mathrm{d}\theta \not\equiv 0 \mod \theta$ this in turn implies that μ is a constant and $C = D = \mu$. Since C, D are constant we must also have

$$\sigma^*(p\mathrm{d}x) = Cp\mathrm{d}x + \mathrm{d}(f(x, p)), \quad \sigma^*(q\mathrm{d}y) = Dq\mathrm{d}y + \mathrm{d}(g(y, q))$$

with f, g certain functions. Then we can calculate

$$\sigma^*(\mathrm{d}z) = \sigma^*(\theta) + \sigma^*(p\mathrm{d}x) + \sigma^*(q\mathrm{d}y) = \mu\mathrm{d}z + \mathrm{d}(f(x, p) + g(y, q)).$$

From this it follows

$$\begin{aligned}
&\sigma^*(x) = a(x, p), \quad \sigma^*(p) = b(x, p),\\
&\sigma^*(y) = m(y, q), \quad \sigma^*(q) = n(y, q),\\
&\sigma^*(z) = \text{const} + \mu z + f(x, p) + g(y, q).
\end{aligned}$$

The condition that $\sigma^*(\theta) = \mu\theta$ implies that we have four relations between the unknown functions a, b, m, n, f, g

$$\begin{aligned}
f_p - ba_p = 0, \quad f_x - a_x b + \mu p = 0,\\
g_q - nm_q = 0, \quad g_y - m_y n + \mu q = 0.
\end{aligned} \tag{9.13}$$

The preservation of Ψ^1 and Ψ^2 implies the equations

$$a_x b_p - a_p b_x = \mu, \quad m_y n_q - m_q n_y = \mu. \tag{9.14}$$

Note that the last two equations are precisely the compatibility equations for the system (9.13) as a system of partial differential equations for the two unknown functions f and g. For arbitrary functions a, m we can first solve the system (9.14). The solutions b, n depend on two more arbitrary functions of one variable. Then we can solve the system (9.13) since the compatibility equations for the system are satisfied. The functions a, b, m, n, f, g and μ together define a (local) symmetry of the wave equation.

Let us give a geometric interpretation of the construction above. The contact form θ and the two characteristic forms Ψ^1 and Ψ^2 together determine the two characteristic systems $\mathcal{W}_+$ and $\mathcal{W}_-$ (see Section 5.5.1). The contact form θ is determined up to a constant. This implies there is a Reeb vector field Z unique up to a scalar constant. This Reeb vector field determines a unique foliation of the equation manifold M. The Reeb vector field is contained in the rank 1 bundle $\mathcal{Z} = \mathcal{W}_+ \cap \mathcal{W}_-$. Let $\pi : M \to B$ be the projection onto the quotient space of M by the foliation. The characteristic systems project to integrable bundles $\tilde{\mathcal{W}}_+$, $\tilde{\mathcal{W}}_-$ on B and give B a direct product structure $B_1 \times B_2$.

$$\begin{array}{c} M \\ \Big\downarrow{\scriptstyle \pi_1 \times \pi_2} \\ B_1 \times B_2 = M/\mathcal{Z} \end{array}$$

The Reeb vector field is a symmetry of θ and hence $\mathrm{d}\theta$ projects down to a 2-form Ω on B that is determined up to a scalar factor. The non-degeneracy of the contact structure implies that Ω restricts on each component B_j to a volume form Ω_j.

Every contact symmetry of the wave equation must preserve the fibers of the projection π and induce a map on B_1 and B_2 that preserves Ω_1 and Ω_2, respectively. We can now construct all local contact symmetries of the wave equation in three steps.

1. Choose a constant scale factor μ.
2. Choose two (local) diffeomorphisms ϕ_1, ϕ_2 of B_1 and B_2, respectively, that scale the volume forms by a factor μ.
3. Lift the map $\phi = \phi_1 \times \phi_2$ on B to a (local) diffeomorphism on M. Let $\tilde{M}$ be a copy of M. We will lift the graph of ϕ to an integral manifold of $M \times \tilde{M}$ that is (locally) the graph of a contact symmetry. A contact symmetry is given by a 5-dimensional integral manifold for the exterior differential system on $M \times \tilde{M}$ generated by $\Theta = \theta - \tilde{\theta}$ and $\Psi^1 - \tilde{\Psi}^1$ with independence condition given by $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}p \wedge \mathrm{d}q$. The graph of ϕ is a submanifold of dimension four in $B \times \tilde{B}$. Let $S = (\pi_1 \times \pi_2)^{-1}(\mathrm{gr}(\phi))$ be the inverse image of this graph. Then S is a submanifold of dimension six in $M \times \tilde{M}$. The form Θ is not closed on $M \times \tilde{M}$, but the pullback of Θ to S is closed. Hence on the manifold S we have a closed 1-form $\Theta|_S$. The integral surfaces of this closed form can be found using the Frobenius theorem and dependent on one constant c. The

integral surfaces are of dimension 5 and are integral surfaces of the exterior differential system generated by $\Theta = \theta - \tilde{\theta}$ and $\Psi^1 - \tilde{\Psi}^1$ and hence define contact transformations. This choice of constant c corresponds to the contact transformation $z \mapsto z + c$, i.e., translation in the z coordinate.

As an example we will construct the Legendre transformation. In the local coordinates $m = (x, y, z, p, q)$ we have $\mathcal{Z} = \operatorname{span}(\partial_z)$, $\pi_1 : M \to B_1 = \mathbb{R}^2 : m \mapsto (x, p)$ and $\pi_2 : M \to B_2 = \mathbb{R}^2 : m \mapsto (y, q)$. Introduce coordinates $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}$ on a copy $\tilde{M}$ of the equation manifold M.

1. We choose scale factor $\mu = 1$.
2. As volume-preserving diffeomorphisms we take $\phi_1 : \mathbb{R}^2 \to \mathbb{R}^2 : (x, p) \mapsto (-p, x)$ and $\phi_2 : \mathbb{R}^2 \to \mathbb{R}^2 : (y, q) \mapsto (-q, y)$.
3. On the pullback S we can introduce coordinates $x, y, z, p, q, \tilde{z}$. The inclusion $S \to M \times \tilde{M}$ is given by $(x, y, z, p, q, \tilde{z}) \mapsto ((x, y, z, p, q), (-p, -q, \tilde{z}, x, y))$. The pullback of Θ equals
$$\begin{aligned}\Theta|_S &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y - (\mathrm{d}\tilde{z} + x\mathrm{d}p + y\mathrm{d}q)\\ &= \mathrm{d}z - \mathrm{d}(xp) - \mathrm{d}(yq) - \mathrm{d}\tilde{z}.\end{aligned}$$
The integral manifolds are given by $\tilde{z} = z - px - qy + c$. The choice $c = 0$ gives the Legendre transformation. $\oslash$

Example 9.3.11 (Contact symmetry group using Cartan-Kähler). There is another method to analyze the "dimension" of the space of contact symmetries. We formulate the contact symmetry problem for the wave equation as an equivalence problem on the first order contact manifold P. We introduce the coframing $\theta^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y$, $\theta^1 = \mathrm{d}p$, $\theta^2 = \mathrm{d}q$, $\theta^3 = \mathrm{d}x$, $\theta^4 = \mathrm{d}y$. To this coframing we have to associate the structure group G consisting of the matrices
$$g = \begin{pmatrix} c_0 & 0 & 0 & 0 & 0 \\ 0 & c_{11} & 0 & c_{13} & 0 \\ 0 & 0 & c_{22} & 0 & c_{42} \\ 0 & c_{31} & 0 & (c_0 + c_{31}c_{13})/c_{11} & 0 \\ 0 & 0 & c_{42} & 0 & (c_0 + c_{42}c_{24})/c_{22} \end{pmatrix}.$$
The group maps θ^0 to a multiple of θ^0 and preserves the ideals $\operatorname{span}(\mathrm{d}x, \mathrm{d}p)$ and $\operatorname{span}(\mathrm{d}y, \mathrm{d}q)$. Also the structure equation $\mathrm{d}\theta^0 \equiv \theta^3 \wedge \theta^1 + \theta^4 \wedge \theta^2 \mod \theta^0$ is preserved. We have formulated the structure of the wave equation in the form of an equivalence problem and can apply the method of equivalence to determine the symmetries. We write $\omega = g^{-1}\theta$ for the lifted coframe on $P \times G$. The tableau associated to this system has the form
$$\Pi = \begin{pmatrix} \gamma_0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{11} & 0 & \gamma_{13} & 0 \\ 0 & 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & \gamma_{31} & 0 & \gamma_0 - \gamma_{11} & 0 \\ 0 & 0 & \gamma_{42} & 0 & \gamma_0 - \gamma_{22} \end{pmatrix}.$$

The Cartan characters are $s_1 = 5$, $s_2 = 2$, $s_3 = s_4 = s_5 = 0$. The dimension of the first prolongation is 8, so the system is not in involution. We write the structure equations as

$$\mathrm{d}\omega = -\tilde{\gamma} \wedge \omega + T(\omega \wedge \omega).$$

Here $\tilde{\gamma}$ is the connection form and T respresents the torsion of the system. We can parameterize the possible connection forms by

$$\tilde{\gamma} = \gamma + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & h_1\omega^1 + h_3\omega^3 & 0 & h_3\omega^1 + h_5\omega^3 & 0 \\ 0 & 0 & h_2\omega^2 + h_4\omega^4 & 0 & h_4\omega^2 + h_6\omega^4 \\ 0 & h_7\omega^1 - h_1\omega^3 & 0 & -h_1\omega^1 - h_3\omega^3 & 0 \\ 0 & 0 & h_8\omega^2 - h_2\omega^4 & 0 & -h_2\omega^2 - h_4\omega^4 \end{pmatrix},$$

where γ is the left-invariant Maurer-Cartan form of the Lie group G.

Let H be the 8-dimensional abelian group with coordinates $h_1, \ldots, h_8$. Consider the system ω, $\tilde{\gamma}$ on the manifold $P \times G \times H$. The structure equations are

$$\mathrm{d}\omega = -\tilde{\gamma} \wedge \omega + T(\omega \wedge \omega),$$

$$\mathrm{d}\tilde{\gamma} = -\tilde{\gamma} \wedge \tilde{\gamma} +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \eta^1 \wedge \omega^1 + \eta^3 \wedge \omega^3 & 0 & \eta^3 \wedge \omega^1 + \eta^5 \wedge \omega^3 & 0 \\ 0 & 0 & \eta^2 \wedge \omega^2 + \eta^4 \wedge \omega^4 & 0 & \eta^4 \wedge \omega^2 + \eta^6 \wedge \omega^4 \\ 0 & \eta^7 \wedge \omega^1 - \eta^1 \wedge \omega^3 & 0 & -\eta^1 \wedge \omega^1 - \eta^3 \wedge \omega^3 & 0 \\ 0 & 0 & \eta^8 \wedge \omega^2 - \eta^2 \wedge \omega^4 & 0 & -\eta^2 \wedge \omega^2 - \eta^4 \wedge \omega^4 \end{pmatrix}$$

where $\eta^j \equiv \mathrm{d}h^j$ modulo terms $\tilde{\gamma}$ and ω. There is no torsion and the Cartan characters are $s_1 = 6$, $s_2 = 2$, $s_3 = 0$. The dimension of the first prolongation is 10. Since $s_1 + 2s_2 = 10$, the system is in involution and from the Cartan-Kähler theorem it follows the general contact symmetry depends on 2 functions of 2 variables. $\oslash$

Example 9.3.12 (Pseudosymmetries for the wave equation). In this example we will apply the method described in Section 9.3 to the wave equation. We start with the following adapted coframing for the wave equation

$$\begin{aligned} \theta^0 &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y, \\ \theta^1 &= \mathrm{d}p - r\mathrm{d}x, \\ \theta^2 &= \mathrm{d}q - t\mathrm{d}y, \\ \omega^3 &= \mathrm{d}x, \quad \omega^4 = \mathrm{d}y, \\ \omega^5 &= \mathrm{d}r, \quad \omega^6 = \mathrm{d}t. \end{aligned}$$

The structure equations for this coframe are

$$\begin{aligned}
\mathrm{d}\theta^0 &= -\theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4,\\
\mathrm{d}\theta^1 &= -\omega^5 \wedge \omega^3,\\
\mathrm{d}\theta^2 &= -\omega^6 \wedge \omega^4,\\
\mathrm{d}\omega^3 &= 0, \quad \mathrm{d}\omega^5 = 0,\\
\mathrm{d}\omega^4 &= 0, \quad \mathrm{d}\omega^6 = 0.
\end{aligned}$$

We introduce new contact forms $\tilde{\theta}^{1,2} = \theta^{1,2} + p_{1,2}\theta^0$. We want to find all functions p_1, p_2 such that $\tilde{\theta}^1$ and $\tilde{\theta}^2$ decompose modulo $\tilde{\theta}^1, \tilde{\theta}^2$. The condition is given by equation (9.11), which reduces in this case to

$$\begin{aligned}
(\mathrm{d}p_1 + p_1 p_2 \omega^4) &\equiv 0 \mod \theta^0, \theta^1, \theta^2, \omega^3, \omega^5,\\
(\mathrm{d}p_2 + p_1 p_2 \omega^3) &\equiv 0 \mod \theta^0, \theta^1, \theta^2, \omega^4, \omega^6.
\end{aligned}$$

In local coordinates these two equations are equivalent to the system of equations $\partial p_1/\partial\omega^4 = -p_1 p_2$, $\partial p_1/\partial\omega^6 = 0$, $\partial p_2/\partial\omega^3 = -p_1 p_2$, $\partial p_2/\partial\omega^5 = 0$, or equivalently

$$\begin{aligned}
\frac{\partial p_1}{\partial y} + q\frac{\partial p_1}{\partial z} + t\frac{\partial p_1}{\partial q} &= -p_1 p_2, \quad \frac{\partial p_1}{\partial t} = 0,\\
\frac{\partial p_2}{\partial x} + p\frac{\partial p_2}{\partial z} + r\frac{\partial p_2}{\partial p} &= -p_1 p_2, \quad \frac{\partial p_2}{\partial r} = 0.
\end{aligned}$$

There seems to be no exact method to solve these equations. In the generic situation ($p_1, p_2 \neq 0$) the solutions of this system depend on 2 functions of two variables. The calculations are in Example 9.3.13 below.

However, we can find a special 3-parameter family of solutions by taking $p_2 = 0$. Then the most general solution for p_1 is just $p_1 = a(x, p, r)$. We find a new coframing

$$\begin{aligned}
\tilde{\theta}^0 &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y,\\
\tilde{\theta}^1 &= \mathrm{d}p - r\mathrm{d}x + a(x, p, r)(\mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y),\\
\tilde{\theta}^2 &= \mathrm{d}q - t\mathrm{d}y.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{d}\tilde{\theta}^1 &\equiv \left(\omega^3 - a_r\theta^0\right) \wedge \left(\omega^5 + (a_x + ra_p - a^2)\theta^0\right) \mod \tilde{\theta}^1, \tilde{\theta}^2\\
\mathrm{d}\tilde{\theta}^2 &= \omega^4 \wedge \omega^6.
\end{aligned}$$

So we should adapt our coframing and redefine $\omega^3, \omega^5, \omega^4, \omega^6$ to

$$\begin{aligned}
\tilde{\omega}^3 &= \omega^3 - a_r\tilde{\theta}^0,\\
\tilde{\omega}^5 &= \omega^5 + (a_x + ra_p - a^2)\tilde{\theta}^0,\\
\tilde{\omega}^4 &= \omega^4, \quad \tilde{\omega}^6 = \omega^6.
\end{aligned}$$

We have found a new adapted coframing that defines a pseudosymmetry. The vector field dual to $\tilde{\theta}^0$ is

$$V = (1 + pa_r)\partial_z + (ra_r - a)\partial_p - (a_x + ra_p - a^2)\partial_r + a_r\partial_x. \tag{9.15}$$

Let us check that the vector field V is a pseudosymmetry of the system. The characteristic systems for the wave equation are given by

$$\begin{aligned}\mathcal{F} &= \operatorname{span}\left(F_1 = \partial_x + p\partial_z + r\partial_p, G_1 = \partial_r\right),\\ \mathcal{G} &= \operatorname{span}\left(G_1 = \partial_y + q\partial_z + t\partial_q, G_2 = \partial_t\right).\end{aligned}$$

We calculate the Lie brackets of ψV with representative elements from the characteristic bundles.

$$\begin{aligned}[\psi V, F_1] &\equiv -(a\psi + F_1(\psi))V \mod \mathcal{F},\\ [\psi V, F_2] &\equiv -F_2(\psi)V \mod \mathcal{F},\\ [\psi V, G_1] &\equiv -G_1(\psi)V \mod \mathcal{G},\\ [\psi V, G_2] &\equiv -G_2(\psi)V \mod \mathcal{G}.\end{aligned}$$

Hence V is a pseudosymmetry. We can only scale V to a true symmetry if we can solve the system

$$G_1(\psi) = G_2(\psi) = F_2(\psi) = 0, \quad F_1(\psi) = -a\psi.$$

This implies $\psi = \psi(x, p)$ with $\psi_x + r\psi_p = -a\psi$. A necessary condition for this to be solvable is that a is linear in r and hence of the form $a = \alpha + \beta r$. When substituted into the equation this yields the system

$$\psi_x + \alpha\psi = 0, \quad \psi_p + \beta\psi = 0.$$

From Example 1.2.20 it follows that this system has non-zero solutions if and only if $\alpha_x = \beta_p$. So we have found a family of pseudosymmetries depending on 1 function of 3 variables for which most pseudosymmetries cannot be scaled to a true symmetry.

We conclude with two final remarks. First note that by taking a pseudosymmetry of the form $p_1 = p_1(x, p, r)$, $p_2 = 0$ the projected system will be Darboux integrable. It has 3 invariants for the first characteristic system and 2 or 3 for the second. Second note that the first order part of V for $a = \alpha + r\beta$ corresponds precisely the the pseudosymmetry V described in Example 9.2.19. ⊘

Example 9.3.13 (Prolonging to involution). In the previous example we arrived at a system of partial differential equations (9.3.12) that we could not solve explicitly. In this example we will formulate the system as an exterior differential system and apply the Cartan-Kähler theorem. Let M be the equation manifold of the wave equation and let $N = M \times \mathbb{R}^2$ On N we use coordinates x, y, z, p, q, r, t, p_1, p_2. We have a basis of differential forms on this manifold given by

$$\begin{aligned}
\theta^0 &= \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y,\\
\theta^1 &= \mathrm{d}p - r\mathrm{d}x,\\
\theta^2 &= \mathrm{d}q - t\mathrm{d}y,\\
\omega^3 &= \mathrm{d}x, \qquad \omega^4 = \mathrm{d}y,\\
\omega^5 &= \mathrm{d}r, \qquad \omega^6 = \mathrm{d}t,\\
\pi^1 &= \mathrm{d}p_1, \quad \pi^2 = \mathrm{d}p_2.
\end{aligned}$$

We let $\omega^0 = \theta^0$, $\omega^1 = \theta^1 + p_1\theta^0$, $\omega^2 = \theta^2 + p_2\theta^0$ and $\Omega = \omega^0 \wedge \omega^1 \wedge \omega^2$. The functions p_1, p_2 that determine pseudosymmetries determine 7-dimensional integral manifolds of the exterior differential ideal $\mathcal{I}$ generated by two 5-forms

$$\Psi^1 = (\mathrm{d}\omega^1)^2 \wedge \omega^1 \wedge \omega^2, \quad \Psi^2 = (\mathrm{d}\omega^2)^2 \wedge \omega^1 \wedge \omega^2.$$

Conversely, every 7-dimensional integral manifold of $\mathcal{I}$ that satisfies the independence condition

$$\Omega \wedge \omega^3 \wedge \omega^4 \wedge \omega^5 \wedge \omega^6 \neq 0 \tag{9.16}$$

determines a pair of functions p_1, p_2 and hence a pseudosymmetry.

The properties of the integral manifolds can be found by applying the Cartan-Kähler theorem. Before we can apply this theorem we first have to eliminate torsion. We also have to prolong the system several times so that the system is in involution. Prolonging the system has the additional benefit that the new system is a linear Pfaffian system.

To prolong the system we have the consider the manifold of all integral planes of dimension 7. The integral planes that satisfy the independence condition are parameterized by

$$\pi^1 = p_{1,j}\omega^j, \quad \pi^2 = p_{2,j}\omega^j.$$

Substituting these into the forms Ψ^1, Ψ^2, $\mathrm{d}\Psi^1$, $\mathrm{d}\Psi^2$ yields 6 equations; these equations determine the integral elements.

$$\begin{aligned}
p_{1,4} &= p_1 p_2, \quad p_{1,6} = 0,\\
p_{1,2} &= p_1 p_{2,6} + p_2 p_{2,6},\\
p_{2,3} &= p_1 p_2, \quad p_{2,5} = 0,\\
p_{2,1} &= p_2 p_{1,5} + p_1 p_{2,5}.
\end{aligned} \tag{9.17}$$

We restrict to the submanifold defined by the equations (9.17). This submanifold is locally equivalent to $M \times \mathbb{R}^2 \times \mathbb{R}^8$. We have the original manifold $N = M \times \mathbb{R}^2$ with the variables p_1, p_2 and the new coordinates $p_{1,0}$, $p_{1,1}$, $p_{1,3}$, $p_{1,4}$, $p_{2,0}$, $p_{2,2}$, $p_{2,5}$, $p_{2,6}$. The integral manifolds of the original system $(N, \mathcal{I})$ are lifted to integral manifolds of the system on $N \times \mathbb{R}^8$ defined by the pullbacks of

$$\alpha^1 = \pi_1 - p_{1,j}\omega^j, \quad \alpha^2 = \pi_2 - p_{2,j}\omega^j.$$

The structure equations for α^1 and α^2 are

$$\begin{aligned} \mathrm{d}\alpha^1 &= \tau^{1,0} \wedge \omega^0 + \tau^{1,1} \wedge \omega^1 + p_1 \tau^{2,6} \wedge \omega^2 + \tau^{1,3} \wedge \omega^3 + \tau^{1,5} \wedge \omega^5, \\ \mathrm{d}\alpha^2 &= \tau^{2,0} \wedge \omega^0 + \tau^{2,2} \wedge \omega^2 + p_2 \tau^{1,6} \wedge \omega^1 + \tau^{2,4} \wedge \omega^4 + \tau^{2,6} \wedge \omega^6. \end{aligned} \tag{9.18}$$

Here we have written

$$\begin{aligned} \tau^{1,0} &= \mathrm{d}p_{1,0} - (p_2 p_{1,0} + p_1 p_{2,0})\omega^4 - (p_{2,6} p_{1,0})\omega^2, \\ \tau^{1,1} &= \mathrm{d}p_{1,1} - (p_{2,6} p_{1,1} + p_1 p_{1,5} p_{2,6})\omega^2 - (p_2 p_{1,1} + p_1 p_2 p_{1,5})\omega^4, \\ \tau^{1,3} &= \mathrm{d}p_{1,3} - (p_{1,0})\omega^1 - ((p_1)^2 p_{2,6} + p_{1,3} p_{2,6})\omega^2 \\ &\qquad - (p_2 p_{1,3} + (p_1)^2 p_2)\omega^4 - (2p_{1,1} - p_{2,0}/p_2)\omega^5, \\ \tau^{1,5} &= \mathrm{d}p_{1,5} - (p_{2,6} p_{1,5})\omega^2 - (-p_{1,1} + p_{2,0}/p_2)\omega^3 - (p_2 p_{1,5})\omega^4. \end{aligned}$$

The expressions for $\tau^{2,0}$, $\tau^{2,2}$, $\tau^{2,4}$ and $\tau^{2,6}$ are similar to the expressions above since the entire system is symmetric in the characteristic systems. By interchanging the pairs $(1, 2)$, $(3, 4)$ and $(5, 6)$ we find the correct expressions. For example

$$\tau^{2,0} = \mathrm{d}p_{2,0} - (p_1 p_{2,0} + p_2 p_{1,0})\omega^3 - (p_{1,5} p_{2,0})\omega^1.$$

In the calculations below an expression τ^* will always be equal to $\mathrm{d}p_*$ modulo some terms ω^k. We will not write down the full expressions since these are complicated and are not necessary to understand the results.

To the linear Pfaffian system generated by α^1 and α^2 corresponds the tableau

$$\begin{pmatrix} \tau^{1,0} & \tau^{1,1} & p_1 \tau^{2,6} & \tau^{1,3} & 0 & \tau^{1,5} & 0 \\ \tau^{2,0} & p_2 \tau^{1,5} & \tau^{2,2} & 0 & \tau^{2,4} & 0 & \tau^{2,6} \end{pmatrix}.$$

We assume from here on that $p_1 \neq 0$ and $p_2 \neq 0$. The Cartan characters are $s_1 = s_2 = s_3 = s_4 = 2$, $s_5 = s_6 = s_7 = 0$. The dimension of the first prolongation is 16. Since $s_1 + 2s_2 + 3s_3 + 4s_4 = 20 \geq 16$, Cartan's test is not satisfied and the system is not in involution.

We continue with a of the system. This turns out to be the most efficient way to arrive at a system in involution, of course a full prolongation would lead to the same result. We expand the variables $p_{1,5}$, $p_{2,6}$ and add the new variables $p_{1,5,j}$, $p_{2,6,j}$, $j = 0, \ldots, 6$. We define the contact forms

$$\alpha^{1,5} = \mathrm{d}p_{1,5} - p_{1,5,j}\omega^j, \quad \alpha^{2,6} = \mathrm{d}p_{2,6} - p_{2,6,j}\omega^j.$$

We are looking for integral manifolds of the system generated by α^1, α^2, $\alpha^{1,5}$ and $\alpha^{2,6}$. First we have to analyze and absorb any torsion present. For example we have

$$\begin{aligned} \mathrm{d}\alpha^1 \equiv{} & (p_{1,0} - p_1 p_{2,6,4} - p_1 p_{2,2})\,\omega^2 \wedge \omega^4 \\ & + p_1 p_{2,6,6}\,\omega^2 \wedge \omega^6 \quad \mathrm{mod}\ \omega^0, \omega^1, \omega^3, \omega^5. \end{aligned}$$

Both terms in the equation above represent intrinsic torsion. We can eliminate this torsion by restricting to the submanifold defined by $p_{2,6,6} = 0$, $p_{1,0} = p_1 p_{2,6,4,} + p_1 p_{2,2}$. We can continue in this way and by analyzing the structure equations for α^1, α^2 we find 10 equations in total. We eliminate the torsion by restricting to the submanifold defined by

$$\begin{aligned}
p_{1,5,5} &= 0, & p_{2,6,6} &= 0,\\
p_{1,0} &= p_1 p_{2,6,4,} + p_1 p_{2,2}, & p_{2,0} &= p_2 p_{1,5,3} + p_2 p_{1,1},\\
p_{1,5,6} &= 0, & p_{2,6,5} &= 0,\\
p_{1,5,4} &= p_2 p_{1,5}, & p_{2,6,3} &= p_1 p_{2,6},\\
p_{1,5,2} &= p_{1,5} p_{2,6} + p_1 p_{2,6,5}, & p_{2,6,1} &= p_{1,5} p_{2,6} + p_2 p_{1,5,6}.
\end{aligned}$$

We continue with the structure equations for $\alpha^{1,5}, \alpha^{2,6}$. We eliminate

$$p_{1,5,0} = p_{1,5}(p_{2,2} - p_{2,6,4}), \quad p_{2,6,0} = p_{2,6}(p_{1,1} - p_{1,5,3}).$$

At this moment we are dealing with a system on a 19-dimensional manifold. The differential ideal is generated by four 1-forms and we are looking for 7-dimensional integral manifolds. The tableau is of the form

$$\begin{pmatrix}
p_1(\tau^{2,2} - \tau^{2,6,4}) & \tau^{1,1} & 0 & \tau^{1,3} & 0 & 0 & 0\\
p_2(\tau^{1,1} - \tau^{1,5,3}) & 0 & \tau^{2,2} & 0 & \tau^{2,4} & 0 & 0\\
p_{1,5} p_1(\tau^{2,2} - \tau^{2,6,4}) & \tau^{1,5,1} & 0 & \tau^{1,5,3} & 0 & 0 & 0\\
p_{2,6} p_2(\tau^{1,1} - \tau^{1,4,3}) & 0 & \tau^{2,6,2} & 0 & \tau^{2,6,4} & 0 & 0
\end{pmatrix}.$$

The Cartan characters are $s_1 = s_2 = 4$, $s_3 = s_4 = s_5 = s_6 = s_7 = 0$. The dimension of the first prolongation is 8. Since $s_1 + 2s_2 = 12 > 8$, Cartan's test is not satisfied and the system is not in involution.

We prolong the system again. We introduce

$$\begin{aligned}
\alpha^{1,1} &= \mathrm{d}p_{1,1} - p_{1,2,j}\omega^j, \quad \alpha^{2,2} = \mathrm{d}p_{2,1} - p_{2,2,j}\omega^j,\\
\alpha^{1,5,3} &= \mathrm{d}p_{1,5,3} - p_{1,5,3,j}\omega^j, \quad \alpha^{2,6,4} = \mathrm{d}p_{2,6,4} - p_{2,6,4,j}\omega^j.
\end{aligned}$$

Next we restrict to the submanifold defined by

$$\begin{aligned}
p_{1,1,2} &= p_1 p_{1,5} p_{2,6} + p_{1,1} p_{2,6}, \quad p_{1,1,4} = p_2 p_{1,1} + p_1 p_2 p_{1,5},\\
p_{1,1,5} &= p_{1,5,1}, \quad p_{1,1,6} = 0,\\
p_{1,5,3,6} &= 0, \quad p_{1,5,3,5} = p_{1,5,1},\\
p_{1,5,3,1} &= -p_{1,5} p_{2,2} + p_{1,5} p_{2,6,4} + p_{1,1,1},\\
p_{1,5,3,3} &= p_{1,1,3} - p_1 p_{2,2} + p_1 p_{2,6,4},\\
p_{1,1,0} &= p_{1,1} p_{2,2} + p_1 p_{1,5} p_{2,2} + p_1 p_2 p_{1,5} p_{2,6} - p_{2,6,4} p_{1,1} - p_1 p_{2,6,4,1},\\
p_{1,5,3,2} &= p_1 p_{2,6} p_{1,5} + p_{2,6} p_{1,5,3},\\
p_{1,5,3,4} &= p_1 p_2 p_{1,5} + p_2 p_{1,5,3},\\
p_{1,5,3,0} &= p_{2,2} p_{1,5,3} - p_{1,5,3} p_{2,6,4} + p_1 p_{1,5} p_{2,2} + p_1 p_2 p_{2,6} p_{1,5} - p_{1,5} p_{2,6,4,3},
\end{aligned}$$

and the equations obtained from the equations above by using the symmetry of the system. On this submanifold there is no torsion any more. The tableau for the system has the form

$$\begin{pmatrix} 0 & 0 & 0 & \tau^{1,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau^{2,4} & 0 & 0 \\ 0 & \tau^{1,5,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau^{2,6,2} & 0 & 0 & 0 & 0 \\ 0 & \tau^{1,1,1} & 0 & \tau^{1,1,3} & 0 & \tau^{1,5,1} & 0 \\ 0 & 0 & \tau^{2,2,2} & 0 & \tau^{2,2,4} & 0 & \tau^{2,2,6} \\ 0 & \tau^{1,1,1} & 0 & \tau^{1,1,3} & 0 & \tau^{1,5,1} & 0 \\ 0 & 0 & \tau^{2,2,2} & 0 & \tau^{2,2,4} & 0 & \tau^{2,2,6} \end{pmatrix}.$$

The Cartan characters are $s1 = 6$, $s_2 = 2$ and the other characters are zero. The tableau has dimension 8 and the dimension of the first prolongation is 10. Since $s_1 + 2s_2 = 10$, the system is in involution. The general pseudosymmetry of the wave equation depends on 2 functions of 2 variables. ⊘

9.4 Vector pseudosymmetries

Recall that we have defined a vector pseudosymmetry of a distribution $\mathcal{V}$ on M as an integrable distribution $\mathcal{U}$ such that $[\mathcal{U}, \mathcal{V}] \equiv 0 \mod \operatorname{span}(\mathcal{V}, \mathcal{U})$. Because $\mathcal{U}$ is integrable, we can locally always construct the quotient manifold $B = M/\operatorname{span}(\mathcal{U})$. The condition that $\mathcal{U}$ is a vector pseudosymmetry together with the condition that $\mathcal{V} \cap \mathcal{U}$ has constant rank implies that the distribution $\mathcal{V}$ projects to a distribution $\tilde{\mathcal{V}}$ on B.

Example 9.4.1. We have already seen many examples of vector pseudosymmetries in the previous chapters.

- The projections of Monge-Ampère equations $(M, \mathcal{V})$ from the second order equation manifold to the first order contact bundle are vector pseudosymmetries. The vector pseudosymmetry is $\mathcal{U} = C(\mathcal{V}')$. This is a vector pseudosymmetry for the two distributions $\mathcal{V}'_\pm$.

- The projections of hyperbolic first order systems $(M, \mathcal{V})$ to a base manifold with almost product structure (Section 7.1.3). The vector pseudosymmetry is given by the bundle $\mathcal{B}_1$. This is a vector pseudosymmetry for $\mathcal{V}'_\pm$.

  The elliptic first order systems that project to an almost complex structure are also examples of projections generated by vector symmetries if we extend the concept of vector pseudosymmetry in the obvious way to the complexified tangent bundle.

- All Darboux integrable second order equations and all Darboux integrable first order systems are examples as well. For a Darboux integrable equation or system the tangent spaces to the fibers of the Darboux projection form an integrable distribution. This distribution is a vector pseudosymmetry for the characteristic systems.

It is a good exercise to go through one or more of these examples and check that the projections are indeed vector pseudosymmetries for the structures mentioned. $\oslash$

Example 9.4.2. On $M = \mathbb{R}^4$ with coordinates x, y, p, q introduce the distributions

$$\begin{aligned} \mathcal{V} &= \operatorname{span}\left(\partial_x + q\partial_p, \partial_y + \exp(p)\partial_q\right), \\ \mathcal{U} &= \operatorname{span}\left(\partial_p, \partial_q\right). \end{aligned}$$

The distribution $\mathcal{U}$ is integrable and $[\mathcal{V}, \mathcal{U}] \equiv 0 \mod \operatorname{span}(\mathcal{V}, \mathcal{U})$, so $\mathcal{U}$ is a vector pseudosymmetry for $\mathcal{V}$. There are are no pseudosymmetries of $\mathcal{V}$ of the form $\alpha\partial_p + \beta\partial_q$. This proves that the projection generated by the vector pseudosymmetry $\mathcal{U}$ can not be given as the composition of two (pseudo)symmetry projections. $\oslash$

Let $(M, \mathcal{V})$ be a first order system or second order scalar equation. We assume the system $(M, \mathcal{V})$ is hyperbolic. We are interested in integrable distributions $\mathcal{U}$ transversal to $\mathcal{V}$ that are vector pseudosymmetries for both Monge systems of $(M, \mathcal{V})$. The condition that $\mathcal{U}$ is a vector pseudosymmetry for both Monge systems is

$$[\mathcal{U}, \mathcal{V}_\pm] \equiv 0 \mod \operatorname{span}(\mathcal{V}_\pm, \mathcal{U}).$$

Such a distribution locally defines a projection $M \to B$ of M onto a 4-dimensional base manifold such that the Monge systems $\mathcal{V}_\pm$ on M are projected to transversal rank 2 distributions $\mathcal{W}_\pm$ on B. Since B has dimension four, the distributions $\mathcal{W}_\pm$ define on B an almost product structure.

The 2-dimensional integral manifolds of $(M, \mathcal{V})$ are locally in correspondence with the 2-dimensional integral manifolds of the almost product structure on B. This correspondence was already described in the context of the method of Darboux (see Section 8.1.2 for the method of Darboux as a projection method) and in the context of pseudosymmetries.

Note that this is a generalization of the method of Darboux. Whereas for a Darboux integrable equation (or system) the projected almost product structure is integrable, this does not need to happen for a general projection generated by a vector pseudosymmetry. Therefore we can call the vector pseudosymmetries of this type *generalized Darboux projections*.

Definition 9.4.3 (Generalized Darboux projection). Let $(M, \mathcal{V})$ be a hyperbolic first order system or hyperbolic second order equation. A *transversal vector pseudosymmetry* for $(M, \mathcal{V})$ is an integrable distribution $\mathcal{U}$ that is transversal to $\mathcal{V}$ and a vector pseudosymmetry of both Monge systems. The projection generated by $\mathcal{U}$ is called a *generalized Darboux projection*. $\ominus$

Example 9.4.4 (Constant mean curvature surfaces). Consider the equation for *constant mean curvature* surfaces

$$\frac{(1+q^2)r - 2pqs + (1+p^2)t}{(1+p^q+q^2)^{(3/2)}} = C.$$

Here C is a constant equal to the mean curvature of the surface. For constant mean curvature zero the equation is called the *minimal surface equation*. For any value of the mean curvature

C, the equation is an elliptic second order equation. We can therefore write the equation as a manifold M with rank 4 distribution $\mathcal{V}$ and a complex structure on $\mathcal{V}$.

The constant mean curvature equation is translation invariant in the x, y and z directions. This means that we can use $\mathcal{U} = \operatorname{span}(\partial_x, \partial_y, \partial_z)$ as a vector pseudosymmetry for $\mathcal{V}$ and the complex characteristic systems of $\mathcal{V}$ (the Monge systems). The quotient of the equation manifold by $\mathcal{U}$ is a manifold B with an almost complex structure. The projection $\pi : M \to B$ intertwines the complex structures on $\mathcal{V}$ and TB.

For the minimal surface equation the projection is a Darboux projection and the projected almost complex structure is integrable. By taking holomorphic curves in the base manifold B and lifting these curves to M we can parameterize all minimal surfaces. In suitable local coordinates this is precisely the Weierstrass representation of minimal surfaces (see Dierkes et al. [22, Section 3.3]). The constant mean curvature equation for non-zero curvature is not Darboux integrable. This implies that the almost complex structure on the base manifold B is not integrable, and the projection defines a generalized Darboux projection. $\oslash$

The projections of the minimal surface equation and of the constant mean curvature equations have the special property that the distribution $\mathcal{U}$ is spanned by symmetries of the equation.

Suppose we have a hyperbolic second order equation with a 3-dimensional Lie algebra of symmetries. This Lie algebra spans an integrable distribution $\mathcal{U}$ on the equation manifold M for the equation. At points where $\mathcal{U}$ is transversal to the contact distribution of $\mathcal{V}$ this defines vector pseudosymmetry for the equation. The projections generated by vector pseudosymmetries of this type are called *symmetry projections*. Examples of equations with symmetry projections are easy to construct. Any second order equation of the form $F(p, q, r, s, t)$ has the translations as a 3-dimensional symmetry group. The Lie algebra spans an integrable distribution that is transversal to the contact distribution on an open subset. For first order systems invariant that are translation invariant in the x and y directions we can define vector pseudosymmetries and symmetry projections in the same way, except that the distribution $\mathcal{U}$ will have rank two.

In Chapter 10 we will see that the Darboux projections for many Darboux integrable equations or systems are not symmetry projections. In Chapter 11 we will give examples of equations that have a generalized Darboux projection that is neither a Darboux projection nor a symmetry projection.

9.5 Miscellaneous

9.5.1 Integrable extensions

Integrable extensions are in a certain sense the converse to pseudosymmetries. The concept is defined in Bryant and Griffiths [17, p. 659] and in Ivey and Landsberg [43, §6.5]. Below we give some examples of the construction of integrable extensions using conservation laws. But first we explain the basic idea. A *conservation law* for a system of partial differential equations is a 1-form that is closed on the solutions of the system. If we can formulate the system as an exterior differential system $\mathcal{I}$ in a manifold M, then a conservation law is a 1-form α such that $\mathrm{d}\alpha \equiv 0 \mod \mathcal{I}$. For any solution to the system, or equivalently any integral manifold of the exterior differential system, the form α pulls back to a closed form and hence there is a potential v such that $\mathrm{d}v = \alpha$. Note that v is not a potential for the entire system since α is not closed.

Given a conservation law α we can add a pseudopotential v to the system by considering the product manifold $M' = M \times \mathbb{R}$ and the exterior differential system $\mathcal{I}'$ generated by $\mathcal{I}$ and $\mathrm{d}v - \alpha$. Any solution to the original system yields a family of solutions of the new system $(M', \mathcal{I}')$ by integration of the form $\mathrm{d}v - \alpha$. Conversely, any integral manifold of the new system produces an integral manifold of the original system by the projection $M \times \mathbb{R} \to M$. The new system $(M', \mathcal{I}')$ is called the *integrable extension* of $(M, \mathcal{I})$ by the conservation law α.

In the examples below the projection $M' \to M$ is given by the integral curves of the vector field. This vector field is a symmetry or pseudosymmetry for the lifted system.

Example 9.5.1 (Wave equation). Consider the first order wave equation (Example 4.6.5) $u_y = v_x = 0$. We let I be the Pfaffian system dual to the contact distribution. The ideal I is spanned by the two contact forms $\theta^1 = \mathrm{d}u - p\mathrm{d}x$ and $\theta^2 = \mathrm{d}v - s\mathrm{d}y$. A conservation law for this system is given by

$$\alpha = u\mathrm{d}x + v\mathrm{d}y.$$

Indeed, $\mathrm{d}\alpha = \mathrm{d}u \wedge \mathrm{d}x + \mathrm{d}v \wedge \mathrm{d}y = 0 \mod I$. We add the pseudopotential z as a new coordinate together with the 1-form $\mathrm{d}z - u\mathrm{d}x - v\mathrm{d}y$. The lifted system is generated by the three 1-forms θ, θ^1 and θ^2. In coordinates $z, x, y, u, v, p = u_x, s = v_y$ these are given by

$$\begin{aligned} \theta &= \mathrm{d}z - u\mathrm{d}x - v\mathrm{d}y, \\ \theta^1 &= \mathrm{d}u - p\mathrm{d}x, \quad \theta^2 = \mathrm{d}v - s\mathrm{d}y. \end{aligned}$$

The lifted system is precisely the exterior differential system for the wave equation $z_{xy} = 0$.⊘

Example 9.5.2 (Liouville equation). In this example we use a conservation law of the Liouville equation to construct an integrable extension. Then we use the general solution of the Liouville equation to find the general solution of the new system. The contact ideal for the Liouville equation is generated by $\theta^0 = \mathrm{d}z - p\mathrm{d}x - q\mathrm{d}y$ and the characteristic 1-forms are

$$\theta^1 = \mathrm{d}p - r\mathrm{d}x - \exp(z)\mathrm{d}y, \quad \theta^2 = \mathrm{d}q - \exp(z)\mathrm{d}x - t\mathrm{d}y.$$

We consider the two conservation laws defined by

$$\alpha = \exp(z)\mathrm{d}x + (1/2)q^2\mathrm{d}y, \quad \beta = (1/2)p^2\mathrm{d}x + \exp(z)\mathrm{d}y.$$

It is not difficult to check that $\mathrm{d}\alpha = \mathrm{d}\beta = 0$ on solutions of the Liouville equation.

We make an integrable extension using the conservation law α. We add the pseudopotential A and the 1-forms $\mathrm{d}A - \alpha$. The extension has coordinates A, x, y, z, p, q, r, t. For every solution $z(x, y)$ of the Liouville equation we can find integral manifolds of the closed form

$\mathrm{d}A - \alpha$. The lifted system is precisely the prolongation of the first order system for the two functions z, A given by

$$A_x = \exp(z), \quad A_y = (1/2)(z_y)^2. \tag{9.19}$$

The vector field ∂_A is a pseudosymmetry of the prolonged characteristic systems.

For a given function z that satisfies the Liouville equation, the solutions of the system (9.19) are a 1-parameter family of functions $A(x, y)$ that satisfy the differential equation

$$A_{xy} = A_x \sqrt{2A_y}. \tag{9.20}$$

Conversely, for any solution A of the equation above we can define

$$z(x, y) = \log\left(\frac{\partial A(x, y)}{\partial x}\right).$$

Then z is a solution to the Liouville equation.

The general solution to the Liouville equation can be expressed as

$$z(x, y) = \log\left(\frac{2\phi'(x)\psi'(y)}{(\phi(x) + \psi(y))^2}\right),$$

for arbitrary functions ϕ, ψ. The system obtained by substitution of this solution in the equations (9.19) yields the (rather complicated) solution

$$A(x, y) = \frac{(\phi + \psi)\int \psi''/(\psi')^2 \mathrm{d}y - 2\psi' + C(\phi + \psi)}{\phi + \psi},$$

where C is an arbitrary constant of integration.

We note that the equation (9.20) is Darboux semi-integrable. One of the characteristic systems has three invariants, the other system has only one first order invariant. So the integration of this equation by solving ordinary differential equations alone is not very surprising. $\oslash$

9.5.2 Bäcklund transformations

Bäcklund transformations are closely related to pseudosymmetries. We do not develop any general theory here but we will give several examples to illustrate that the concept of a pseudosymmetry can be used to understand Bäcklund transformations.

Whenever we have two different pseudosymmetries of a system we can construct a Bäcklund transformation between the two quotient systems. In Example 9.5.3 below we construct a Bäcklund transformation between two first order systems that are not equivalent under contact transformations.

Example 9.5.3. Consider the wave equation $z_{xy} = 0$. The equation manifold M is 7-dimensional and we can introduce the coordinates x, y, z, p, q, r, t. The two characteristic systems are given by

$$\mathcal{F} = \mathrm{span}\left(\partial_x + p\partial_z + r\partial_p, \partial_r\right), \quad \mathcal{G} = \mathrm{span}\left(\partial_y + q\partial_z + t\partial_q, \partial_t\right).$$

We will look at the projections generated by the symmetry $V_1 = \partial_z$ and the pseudosymmetry $V_2 = \partial_z - p\partial_p - r\partial_r$.

The quotient of M by V_1 we denote by M_1. On M_1 we have coordinates $x, y, u = p, v = q, a = r, b = t$. The characteristic systems project down to

$$\mathcal{F}_1 = \operatorname{span}\big(\partial_x + a\partial_u, \partial_a\big), \quad \mathcal{G}_1 = \operatorname{span}\big(\partial_y + b\partial_v, \partial_b\big).$$

These bundles define precisely the equations $u_y = v_x = 0$.

The quotient of M by V_2 we denote by M_2. On M_2 we can use the 6 invariants of V_2 as coordinates. We choose $x, y, \tilde{u} = p\exp(z), \tilde{a} = (r + p^2)\exp(z), \tilde{v} = q, \tilde{b} = t$. The characteristic systems project down to

$$\mathcal{F}_2 = \operatorname{span}\big(\partial_x + \tilde{a}\partial_{\tilde{u}}, \partial_{\tilde{a}}\big), \quad \mathcal{G}_2 = \operatorname{span}\big(\partial_y + \tilde{u}\tilde{v}\partial_{\tilde{u}} + \tilde{b}\partial_{\tilde{v}}, \partial_{\tilde{b}}\big).$$

These bundles define precisely the first order system $u_y = uv, v_x = 0$. This system is (2, 3)-Darboux integrable.

The two projections together give a Bäcklund transformation from the system M_1 to the system M_2. Starting with a solution $u(x, y), v(x, y)$ of the system M_1 we can construct a potential $w(x, y)$ that satisfies $w_x = u, w_y = v$. This potential is a solution to the wave equation and is unique up to a constant. Then we project this solution to the system M_2. We find $\tilde{u} = u\exp(w), \tilde{v} = v$. We can easily check that the pair $\tilde{u}, \tilde{v}$ is a solution to the system M_2. $\oslash$

Example 9.5.4 (Cole-Hopf transformation). The Cole-Hopf transformation is a transformation between the dissipative version for Burgers' equation

$$u_t = (u_x + u^2)_x$$

and the heat equation

$$v_t = v_{xx}.$$

For any solution u of Burgers' equation there exists a potential w such that $w_x = u$ and $w_t = u_x + u^2$. If we then define $v = \exp(w)$, the function v is a solution to the heat equation. Conversely, for any solution v of the heat equation we can take $(\ln v)_x = v_x/v$ as a solution to Burgers' equation.

Let M be the infinite equation manifold for the heat equation, with internal coordinates $x, y, v, v_x, v_{xx}, \ldots$. Let V be a vector field on M and suppose we want that the quotient of M by V gives the equation manifold of Burgers' equation. Then V must leave $x, t, u = \ln(v)_x = v_x/v$ invariant and also leave $u_{xx}, u_{xxx}, \ldots$ invariant. We write v_j for the j-th order derivative of v with respect to x. It is clear that V must be of the form $V = \sum_n \alpha_n \partial_{v_n}$. We define $\alpha_0 = v$ and calculate the other coefficients order by order. (Making another choice of α_0 would lead to the same results).

$$V(u) = V(v_x/v) = \alpha_1/v - v_x\alpha_0/v^2.$$

Hence $\alpha_1 = v_x \alpha_0 / v = v_x$. Continuing in this way we find quickly that

$$V = v\partial_v + v_1 \partial v_1 + v_2 \partial v_2 + \dots.$$

The vector field V is just the scaling symmetry of the heat equation. This is a true symmetry that is already defined on the second order jet bundle.

Note that one solution to Burgers' equation yields a 1-parameter family of solutions to the heat equation. A solution to the heat equation however yields only one solution to Burgers' equation. For this reason the Cole-Hopf transformation is not always regarded as a Bäcklund transformation. It is an example of a classical symmetry reduction. $\oslash$

Example 9.5.5 (KdV and mKdV). Some classical Bäcklund transformations can be formulated as pseudosymmetries on an infinite jet bundle. The example below is a very famous example of a Bäcklund transformation. Here we formulate the transformation as taking the quotient by a pseudosymmetry defined on an infinite jet bundle.

There is a Bäcklund transformation between the Korteweg-de Vries (KdV) equation and the modified Korteweg-de Vries (mKdV) equation. We take as representations of these two equations

$$u_t = -u_{xxx} - 6uu_x, \tag{KdV}$$

$$v_t = -v_{xxx} - 6(\mu - v^2)v_x. \tag{mKdV}$$

The transformation that maps solutions of the mKdV equation to solutions of the KdV equation is

$$u = v_x - v^2 + \mu. \tag{9.21}$$

The converse transformation is through an integration procedure. For $\mu = 0$ this is the Miura transformation. See Ivey and Landsberg [43, p. 234] or [73] for more details.

We introduce the equation manifold M for the mKdV equation with coordinates x, t, v, $v_x = v_1$, $v_{xx} = v_2, \dots$ and the equation manifold B for the KdV equation with coordinates $x, t, u, u_1, u_2, \dots$. On M we define the total vector fields $X = \partial_x + v_1 \partial_v + v_2 \partial_{v_1} + \dots$ and

$$T = \partial_t + (-v_3 - 6(\mu - v^2)v_x)\partial_v + X(-v_3 - 6(\mu - v^2)v_x)\partial_{v_1} + \dots.$$

These two total vector fields span the contact distribution on the infinite jet bundle. On B we define the total vector fields $X_B = \partial_x + v_1 \partial_v + \dots$ and

$$T_B = \partial_t + (-u_3 - 6uu_x)\partial_u + X_B(-u_3 - 6uu_x)\partial_{u_1} + \dots.$$

The transformation (9.21) defines a projection $\pi : M \to B$. The point $(x, t, v, v_1, \dots)$ on M is mapped to the point $(x, y, u, u_1, \dots)$ on B with $u_n = X^n(v_x - v^2 + \mu)$. Under the projection the total vector fields X and T on M are mapped to the total vector fields X_B and T_B on B, respectively.

We define the vector field $V = \sum_n \alpha_n \partial_{v_n}$ with $\alpha_0 = 1$, $\alpha_1 = 2v$, $\alpha_n = VX^{n-1}(v^2)$. The vector field V is well-defined, one can calculate the terms α_n order by order. The term

α_n only depends on the part of V of order $n-1$. For example $\alpha_1 = V(v^2) = 2v$ and $\alpha_2 = VX(v^2) = V(2vv_1) = 2V(v)v_1 + 2vV(V_1) = 2v_1 + 4v^2$. Note that V is constructed such that $V(v_n - X^{n-1}(v^2)) = VX^{n-1}(v_x - v^2 + \mu) = 0$. This expresses that V leaves u_n invariant for all $n \geq 0$. Since V leaves invariant x, t, u, u_1, $\ldots$, the quotient of M by the integral curves of V gives the transformation $u = v_x - v^2 + \mu$. Hence the projection π from M to B is defined by taking quotient of M by the integral curves of the vector field V. We will show that V is a pseudosymmetry of the contact structure on M in the sense that $[V, X] \equiv 0 \mod V$ and $[V, T] \equiv 0 \mod V$

We claim that $[V, X] = 2vV$ and will proof this order by order. First note that $V = \partial_v + 2v\partial_{v_1} + (2v_1 + 4v^2)\partial_{v_2}$ plus higher order terms. Then

$$[V, X] = (2v\partial_v + \ldots) - (X(2v)\partial_{v_1} + \ldots) = 2v\partial_v + \ldots.$$

So $[V, X] = 2vV$ modulo terms ∂_{v_1} and higher order. Now suppose $[V, X] = 2vV$ up to order m. Then we find

$$\begin{aligned}
[V, X] &= [\sum_n \alpha_n \partial_{v_n}, \partial_x + \sum_n v_{n+1}\partial_{v_n}] \\
&= \sum_n \alpha_{n+1}\partial_{v_n} - X(\alpha_n)\partial_{v_n} \\
&= \sum_n (\alpha_{n+1} - X(VX^{n-1}(v^2)))\partial_{v_n} \\
&= \sum_n (\alpha_{n+1} - VX^n(v^2) - [X, V]X^{n-1}(v^2))\partial_{v_n} \\
&= \sum_n [V, X]X^{n-1}(v^2))\partial_{v_n}.
\end{aligned}$$

Note that $X^{n-1}(v^2)$ is of order $n-1$ in x. Therefore by our induction assumption we have

$$\begin{aligned}
[V, X] &= \sum_{n=1}^{m} 2vVX^{n-1}(v^2))\partial_{v_n} + \ldots \\
&= \sum_{n=1}^{m} 2v\alpha_n\partial_{v_n} + \ldots = 2vV + \ldots,
\end{aligned}$$

where the ellipsis mean order $m+1$ or higher. By induction we have $[V, X] = 2vV$.

By a direct calculation we find

$$VX(v_t) = -4vv_2 + \alpha_2(-6\mu + 6v^2) - 2v_3 - 16v^4 = VT(v^2).$$

Using the commutation relations we then find for all $n \geq 0$

$$VX^n(v_t) = VTX^{n-1}(v^2).$$

We want to prove that $[V, T] = \kappa_t V$ for a differential function κ_t. By calculating the first few terms of $[V, T]$ by hand we quickly find that κ_t must be equal to $(-12v\mu + 4v^3 - 2v_2)$

and that $[V, T] = \kappa_t V$ holds for the lowest order terms. Assume the statement holds up to order m. Using the properties of V found already and the fact that $[X, T] = 0$ we calculate

$$\begin{aligned}
[V, T] &= \sum_n VX^n(v_t)\partial_{v_n} - TVX^{n-1}(v^2)\partial_{v_n} \\
&= \sum_n (VX^n v_t - VTX^{n-1}(v^2) + [V, T]X^{n-1}(v^2))\partial_{v_n} \\
&= \sum_n [V, T]X^{n-1}(v^2)\partial_{v_n} \\
&= \sum_{n=0}^{m} \kappa_t VX^{n-1}(v^2)\partial_{v_n} + \ldots \\
&= \kappa_t V + \ldots \quad \text{(terms of order } m+1 \text{ and higher).}
\end{aligned}$$

We have proved the induction step and hence $[V, T] = \kappa_t V$. The vector field V preserves the total vector fields modulo V itself and hence V is a pseudosymmetry.

The vector fields V cannot be scaled to a symmetry. To make ϕV into a symmetry means solving $2v\phi - X(\phi) = 0$, $\kappa_t\phi - T(\phi) = 0$. This is not possible with a differential function ϕ, i.e., a function depending on a finite number of jet coordinates. We can say that the Bäcklund transformation from the KdV equation to the mKdV equation is an example of a pseudosymmetry projection. ⊘

Example 9.5.6 (Sine-Gordon equation). An auto-Bäcklund transformation is a Bäcklund transformation from a system to the system itself. A classical example of an auto-Bäcklund transformation is that of the Sine-Gordon equation. Consider the system

$$\begin{aligned}
u_x - v_x &= \lambda \sin(u + v), \\
u_y + v_y &= \lambda^{-1} \sin(u - v).
\end{aligned} \tag{9.22}$$

For any solution u, v of the system, the individual functions u and v both satisfy the Sine-Gordon equation

$$u_{xy} = \sin(2u). \tag{9.23}$$

Conversely, for any solution u of the Sine-Gordon equation the system (9.22) reduces to a compatible determined system for v. Hence we can solve v by integration. The transformation from u to v using this procedure is an auto-Bäcklund transformation of the Sine-Gordon equation. We will formulate this transformation in terms of a pseudosymmetry. The system is formulated on a higher order bundle than in the normal presentation. We need this one order higher to be able to define the pseudosymmetries.

The system (9.22) is a hyperbolic first order system. In coordinates $x, y, u, v, a = u_x$, $b = v_y$ the two characteristic systems are given by

$$\begin{aligned}
&\mathrm{span}(\partial_x + a\partial_u + (a - \lambda \sin(u + v))\partial_v + \sin(v)\cos(v)\partial_b, \partial_a), \\
&\mathrm{span}(\partial_y + -(b - \lambda^{-1} \sin(u - v))\partial_u + b\partial_v + \sin(u)\cos(u)\partial_a, \partial_b).
\end{aligned}$$

We prolong this system one time. The prolongation $(M^{(1)}, \mathcal{F}^{(1)}, \mathcal{G}^{(1)})$ is a hyperbolic exterior differential system of class $s = 4$. We introduce two additional coordinates A, B. The prolonged characteristic systems are given by

$$\begin{aligned}\mathcal{F}^{(1)} &= \operatorname{span}\big(\partial_x + a\partial_u + (a - \lambda\sin(u+v))\partial_v \\ &\qquad + \sin(v)\cos(v)\partial_b + A\partial_a + b(2\cos(v)^2 - 1)\partial_B, \partial_A\big), \\ \mathcal{G}^{(1)} &= \operatorname{span}\big(\partial_y + -(b - \lambda^{-1}\sin(u-v))\partial_u + b\partial_v \\ &\qquad + \sin(u)\cos(u)\partial_a + a(2\cos(u)^2 - 1)\partial_A, \partial_B\big).\end{aligned}$$

For the prolonged system we want to find pseudosymmetries. We want the first pseudosymmetry to project onto a system describing the equation $u_{xy} = \sin(2u)$. Using the *ansatz*

$$V = \partial_v + \alpha\partial_b + \beta\partial_B$$

and the condition that V leaves invariant x, y, u, u_x, u_{xx}, we quickly find that we should take

$$\begin{aligned}V_1 = \partial_v &+ -\lambda^{-1}(\cos(u)\cos(v) + \sin(u)\sin(v))\partial_b \\ &- \lambda^{-1}\big(2b(\sin(u)\cos(v) - \cos(u)\sin(v)) - \lambda^{-1}\big)\partial_B.\end{aligned}$$

By direct calculation we find that V_1 is a true pseudosymmetry of the distributions $\mathcal{F}^{(1)}$ and $\mathcal{G}^{(1)}$. In a similar way we find that

$$\begin{aligned}V_2 = \partial_u &+ \lambda(\cos(u)\cos(v) - \sin(u)\sin(v))\partial_a \\ &- \lambda\big(2a(\sin(u)\cos(v) + \cos(u)\sin(v)) - \lambda\big)\partial_A\end{aligned}$$

is a pseudosymmetry of $\mathcal{F}^{(1)}$ and $\mathcal{G}^{(1)}$. The vector field V_2 leaves v and the derivatives of v with respect to y invariant.

The quotient of the 8-dimensional equation manifold $M^{(1)}$ by the integral curves of V_1 is a 7-dimensional manifold with two characteristic bundles. By introducing proper coordinates it is not difficult to see that this manifold is precisely the equation manifold for the Sine-Gordon equation for u. In the same way the quotient of $M^{(1)}$ by V_2 is equal to the equation manifold of the Sine-Gordon equation for v.

The auto-Bäcklund transformation of the Sine-Gordon equation can be described as follows. We start with a solution of the Sine-Gordon equation. We can lift this solution to a 1-parameter family of solutions of the hyperbolic exterior differential system $M^{(1)}$ using the pseudosymmetry V_1. Then projection to the equation manifold of the Sine-Gordon equation using the other pseudosymmetry V_2 yields a 1-parameter family of solutions of the Sine-Gordon equation.

$\oslash$

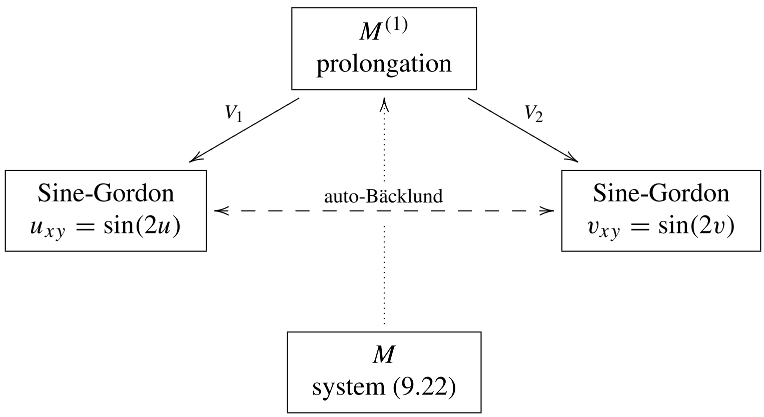

Figure 9.2: Bäcklund transformation for the Sine-Gordon equation.

Chapter 10

Tangential symmetries

We give a geometric construction of the tangential symmetries described by Vassiliou [66]. These tangential symmetries are a geometric realization of the Lie algebras associated to Darboux integrable equations by Vessiot [69, 70]. Our geometric construction makes it possible to generalize a large part of the results to a much more general setting.

10.1 Reciprocal Lie algebras

Let $\mathfrak{g}$ be an n-dimensional Lie algebra and let $\alpha : \mathfrak{g} \to \mathscr{X}(M)$ be an injective representation of $\mathfrak{g}$ in the space of vector fields on M. In the theory to be developed below we will work locally and M will be of the same dimension as $\mathfrak{g}$, so we can think of M as an open subset of $\mathbb{R}^n$. We say the representation is *locally transitive* if $\alpha(\mathfrak{g})$ locally generates as a $C^\infty(M)$-module the space of vector fields $\mathscr{X}(M)$. If $\dim M = \dim \mathfrak{g} = n$, then a transitive representation defines an injective map $\mathfrak{g} \to \mathscr{X}(M)$ and we can identify $\mathfrak{g}$ with its representation as vector fields on M.

A Lie algebra of vector fields on M is a Lie subalgebra of $\mathscr{X}(M)$. For any Lie algebra $\mathfrak{g}$ of vector fields on M we can define the *evaluation map*

$$\mathrm{ev}(\mathfrak{g})_x : \mathfrak{g} \to T_x M : X \mapsto X(x).$$

A Lie algebra $\mathfrak{g}$ of vector fields on M is *locally transitive* if for each point in $x \in M$ the evaluation map $\mathrm{ev}(\mathfrak{g})_x$ is a linear isomorphism from $\mathfrak{g}$ onto $T_x M$.

Example 10.1.1. Let $\mathfrak{g}$ be the 2-dimensional abelian Lie algebra spanned by the two vectors e_1, e_2. A representation of $\mathfrak{g}$ is given by the map $\mathfrak{g} \to \mathscr{X}(\mathbb{R}^2)$

$$e_1 \mapsto \partial_{x^1}, \quad e_2 \mapsto \partial_{x^2}.$$

Let $\mathfrak{aff}(1)$ be the 2-dimensional affine algebra. This algebra is spanned by e_1, e_2 and $[e_1, e_2] = e_2$. A representation is given by

$$e_1 \mapsto \partial_{x^1} - x^2 \partial_{x^2}, \quad e_2 \mapsto \partial_{x^2}. \qquad \oslash$$

We define the *centralizer* of $\mathfrak{g}$ to be the Lie algebra of vector fields that commute with $\mathfrak{g}$.

Theorem 10.1.2. *Let M be a smooth manifold. Let $\mathfrak{g}$ be an n-dimensional locally transitive Lie subalgebra of $\mathscr{X}(M)$. Then the centralizer $\mathfrak{h}$ is an n-dimensional locally transitive Lie algebra of vector fields. The Lie algebra $\mathfrak{g}$ is anti-isomorphic with $\mathfrak{h}$ in the sense that for every point $x \in M$ the linear isomorphism $\alpha_x = (\mathrm{ev}(\mathfrak{h})_x)^{-1} \circ \mathrm{ev}(\mathfrak{g})_x$ is a Lie algebra anti-homomorphism. In particular for all vector fields $X, Y \in \mathfrak{g}$ we have $\alpha_x([X, Y]) = -[\alpha_x(X), \alpha_x(Y)]$.*

Proof. Let $x \in M$. Let G denote the local Lie group of diffeomorphisms which is generated by $\mathfrak{g}$ (see for example Duistermaat and Kolk [28, Section 1.8]). Then $\mathrm{ev}(\mathfrak{g})_x$ is equal to the tangent mapping at the identity element of $g \mapsto g(x)$. Since $\mathrm{ev}(\mathfrak{g})_x$ is bijective we can conclude from the inverse mapping theorem that there is an open neighborhood of the identity element in G such that $\phi : g \mapsto g(x)$ is a diffeomorphism onto an open neighborhood V of x in M. Under this diffeomorphism every element $X \in \mathfrak{g}$ (which is a vector field on M) is identified with the left-invariant vector field X^L on G.

Every $Y \in \mathfrak{h}$ locally defines a vector field $\tilde{Y}$ on G (by the diffeomorphism ϕ). The vector field $\tilde{Y}$ is invariant under the action of $\mathfrak{g}$ and hence is invariant under the infinitesimal right multiplications. But then Y is right-invariant as well and hence $\tilde{Y} = Y^R$. This shows that $\mathfrak{h}$ is generated by the right-invariant vector fields. The right-invariant vector fields define a locally transitive Lie algebra of vector fields on G and hence on M.

Finally we note that $X^L \mapsto X^R$ is a Lie algebra anti-homomorphism from the Lie algebra of left-invariant vector fields to the Lie algebra of right-invariant vector fields and this proves the last part of the theorem. □

If $\mathfrak{g}$ is a locally transitive Lie algebra of vector fields, then the centralizer $\mathfrak{h}$ is locally transitive as well and $\mathfrak{g}$ is the centralizer of $\mathfrak{h}$. We say that $\mathfrak{g}$ and $\mathfrak{h}$ are *reciprocal Lie algebras* and $\mathfrak{h}$ is the reciprocal Lie algebra of $\mathfrak{g}$.

Example 10.1.3 (Reciprocal Lie algebra). Consider the affine Lie algebra $\mathfrak{aff}(1)$, represented by the two vector fields

$$e_1 = \partial_{x^1} - x^2\partial_{x^2}, \quad e_2 = \partial_{x^2}.$$

Then the reciprocal Lie algebra is generated by

$$f_1 = \partial_{x^1}, \quad f_2 = \exp(-x^1)\partial_{x^2}.$$

Note that $[e_1, e_2] = e_2$ and $[f_1, f_2] = -f_2$ so the structure constants for both Lie algebras are related by a minus sign. ⊘

We will end this section with a lemma on reciprocal Lie algebras. We will use this lemma in Section 10.3.1.

Lemma 10.1.4. *Let M be a smooth connected manifold with Lie algebras of vector fields $\mathfrak{g}$, $\mathfrak{h}$. We assume that i) for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ we have $[X, Y] = 0$ and ii) for all $x \in M$ the evaluation maps $\mathrm{ev}(\mathfrak{g})_x : \mathfrak{g} \to T_xM$ and $\mathrm{ev}(\mathfrak{h})_x : \mathfrak{h} \to T_xM$ are surjective. Then for all $x \in M$ the maps $\mathrm{ev}(\mathfrak{g})_x$ and $\mathrm{ev}(\mathfrak{h})_x$ are injective and hence $\mathfrak{g}$, $\mathfrak{h}$ are reciprocal Lie algebras of dimension equal to the dimension of M.*

Proof. Suppose that $X \in \mathfrak{g}$ with $X(x) = 0$. Let H be the local Lie group of local diffeomorphisms with Lie algebra $\mathfrak{h}$. It follows from (ii) that H acts locally transitive on M and from (i) that X is invariant under the action of H. This implies that X is identically zero on a neighborhood of x. It follows that the zero-set of X is open. Since X is continuous the zero-set is closed as well and since we have assumed M is connected $X = 0$. This proves that $\mathrm{ev}(\mathfrak{g})_x$ is injective for all points $x \in M$. The rest of the lemma follows from Theorem 10.1.2. □

10.2 Reciprocal Lie algebras on Lie groups

Theorem 10.2.1 (Lie's Third Fundamental Theorem). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then there is a simply connected Lie group G with Lie algebra equal to $\mathfrak{g}$.*

Proof. See Duistermaat and Kolk [28, Theorem 1.14.3]. □

The Lie algebra $\mathfrak{g} = T_e G$ of a Lie group G is equal to the tangent space of G at the identity element. The Lie brackets on $\mathfrak{g}$ can either be defined using the adjoint action (induced from the conjugation map on G) or using the left-invariant vector fields on G. Using the right-invariant vector fields is also possible, but this leads to a an anti-homomorphic Lie algebra. In our conventions (Section 1.2.1) the map $X \mapsto X^L$ which maps a vector $X \in \mathfrak{g}$ to the left-invariant vector field X^L with $X^L_e = X$ is a Lie algebra homomorphism. The map $X \mapsto X^R$ is a Lie algebra anti-homomorphism. For details on the relation between the different definitions of a Lie algebra see Frankel [34, pp. 402, 486] and Duistermaat and Kolk [28, pp. 2–3, 41].

Theorem 10.2.2. *The left- and right-invariant vector fields on a Lie group commute. The center of the Lie algebra is equal to the vector fields that are both left- and right-invariant.*

Proof. Note that for any Lie group G the left and right multiplication are commutative, i.e., $L_x R_z y = R_z L_x y = xyz$, for all $x, y, z \in G$. From this it follows that

$$T_{yz} L_x \circ T_y R_z = T_{xy} R_z \circ T_y L_x .$$

The left-invariant vector fields act by right translations (Lemma 1.2.1). This implies that the right-invariant vector fields are invariant under the flow generated by left-invariant vector fields and hence the left-invariant and right-invariant vector fields commute. A vector field X that is both left- and right-invariant commutes with both left- and right-invariant vector fields. Hence X is in the center of both $\mathfrak{g}_L$ and $\mathfrak{g}_R$. Conversely, if X is in the center of $\mathfrak{g}_L$, then X^L commutes with all left-invariant vector fields. From the theory of reciprocal Lie algebras it follows that X^L must be a right-invariant vector field. □

The left- and right-invariant vector fields on a Lie group commute and this implies that the left- and right-invariant vector fields are reciprocal Lie algebras.

Example 10.2.3. The left- and right-invariant vector fields on the affine group from Example 1.2.2 are reciprocal.

$$\begin{aligned}
[X_1, Y_1] &= [a\partial_a, a\partial_a + b\partial_b] = 0,\\
[X_1, Y_2] &= [a\partial_a, \partial_b] = 0,\\
[X_2, Y_1] &= [a\partial_b, a\partial_a + b\partial_b] = 0,\\
[X_2, Y_2] &= [a\partial_b, \partial_b] = 0.
\end{aligned}$$

⊘

10.3 Darboux integrable pairs of distributions

In this section we will consider a pair of distributions $\mathcal{F}$, $\mathcal{G}$ on a manifold M. We assume some properties of the bundles and show that this leads to a very rigid structure on the manifold. We denote by $n_{\mathcal{F}}$ and $n_{\mathcal{G}}$ the number of invariants of the bundles $\mathcal{F}$ and $\mathcal{G}$, respectively. As usual we assume all objects defined are of constant rank.

Definition 10.3.1. Let M be an n-dimensional manifold with two distributions $\mathcal{F}, \mathcal{G}$. We call $\mathcal{F}, \mathcal{G} \subset TM$ a *pair of Darboux integrable distributions* on M if $\mathcal{F}, \mathcal{G}$ are of constant rank and the following conditions hold:

- The intersection of $\mathcal{F}$ and $\mathcal{G}$ is empty.
- $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$ has no invariants.
- $[\mathcal{F}, \mathcal{G}] \equiv 0 \mod \mathcal{F} \oplus \mathcal{G}$.
- The number of invariants for each of the bundles is equal to the rank of the other bundle, i.e., $n_{\mathcal{F}} = \mathrm{rank}(\mathcal{G})$ and $n_{\mathcal{G}} = \mathrm{rank}(\mathcal{F})$.

The triple $(M, \mathcal{F}, \mathcal{G})$ is called a *Darboux integrable system* or a *Darboux integrable pair of distributions*. ⊖

Remark 10.3.2. The condition that $\mathcal{V}$ has no invariants is not essential. If $\mathcal{V}$ has p invariants, then the completion of $\mathcal{V}$ is an integrable distribution of codimension p. We can restrict ourselves to the leaves of this distribution and on each leaf the pair $\mathcal{F}$, $\mathcal{G}$ can form a pair of Darboux integrable distributions. For simplicity we will always assume that $\mathcal{V}$ has no invariants. ⊘

Remark 10.3.3. It can happen that $\mathcal{F}$ has more invariants than the rank of $\mathcal{G}$ (or $\mathcal{G}$ more invariants than the rank of $\mathcal{F}$, or both). In this case we can often carry out a construction very similar to the construction to be described below. In particular the Darboux integrability property is preserved. ⊘

Example 10.3.4. Let $M = \mathbb{R}^2$ with coordinates x, y. Then we can take $\mathcal{F}$ to be spanned by the vector field ∂_x and $\mathcal{G}$ spanned by ∂_y. The bundle $\mathcal{F}$ has y as an invariant, the bundle $\mathcal{G}$ has x as an invariant. The triple $(M, \mathcal{F}, \mathcal{G})$ is a Darboux integrable pair of distributions.

This can be generalized. For every direct product $M_1 \times M_2$ the distributions $\mathcal{F} = TM_1$ and $\mathcal{G} = TM_2$ define a Darboux integrable pair of distributions on $M_1 \times M_2$. ⊘

Example 10.3.5. Consider the equation manifold associated to the Liouville equation. On this manifold we have two natural distributions defined by the characteristic systems. In coordinates x, y, z, p, q, r, t we have

$$\begin{aligned}\mathcal{F} &= \operatorname{span}\left(\partial_x + p\partial_z + r\partial_p + \exp(z)\partial_q + q\exp(z)\partial_t,\, \partial_r\right),\\ \mathcal{G} &= \operatorname{span}\left(\partial_x + q\partial_z + \exp(z)\partial_p + t\partial_q + p\exp(z)\partial_r,\, \partial_t\right).\end{aligned}$$

The bundle $\mathcal{F}$ has two invariants $y, t - q^2/2$ and the bundle $\mathcal{G}$ has two invariants $x, r - p^2/2$. The bundles $\mathcal{F}$ and $\mathcal{G}$ form a pair of Darboux integrable distributions. All Darboux integrable equations provide examples of pairs of Darboux integrable distributions. ⊘

Given a pair of Darboux integrable distributions $\mathcal{F}$, $\mathcal{G}$ there is a natural projection onto the space of invariants. The completions of $\mathcal{F}$ and $\mathcal{G}$ are integrable and hence they define a foliation of M of codimension $n_{\mathcal{F}}$ and $n_{\mathcal{G}}$, respectively.

Definition 10.3.6. Locally define B_1 to be the quotient of M by the completion of $\mathcal{G}$ and B_2 to be the quotient of M by the completion of $\mathcal{F}$. Let π_1 and π_2 be the projection of B on B_1 and B_2, respectively. The projection $\pi = \pi_1 \times \pi_2 : M \to B = B_1 \times B_2$ is a natural projection onto the manifold $B = B_1 \times B_2$ of dimension $n_{\mathcal{G}} + n_{\mathcal{F}}$. We call such a projection a *Darboux projection.* ⊖

The tangent spaces to the fibers of the Darboux projection are equal to the integrable distribution $\mathcal{Z} = \mathcal{F}^{\mathrm{compl}} \cap G^{\mathrm{compl}}$. We write $\mathfrak{z}$ for the Lie algebra of vector fields tangent to the projection. The vector fields $\mathfrak{z}$ are precisely the vector fields in the distribution $\mathcal{Z}$. The functions that are invariants of $\mathcal{F}$ are functions in $\pi_1^*(C^\infty(B_1))$. The functions that are invariants of $\mathcal{G}$ are functions in $\pi_2^*(C^\infty(B_2))$.

Lemma 10.3.7. *The distributions $\mathcal{F}$ and $\mathcal{G}$ project onto B. The image of $\mathcal{F}$ is equal to the tangent space of $B_1 \times \{\,\mathrm{pt}\,\}$ and the image of $\mathcal{G}$ is equal to the tangent space of $\{\,\mathrm{pt}\,\} \times B_2$.*

Proof. The last condition in Definition 10.3.1 implies that $\dim B = \operatorname{rank} \mathcal{V}$. The together with the fact that $\mathcal{V}$ has no invariants implies that the projection of $\mathcal{V}$ is onto TB. Since $\mathcal{F}$ is contained in $\mathcal{F}^{\mathrm{compl}}$ and B_2 is defined locally as the foliation of M by the leaves of the completion of $\mathcal{F}$, the projection of vectors in $\mathcal{F}$ is contained in the tangent space to $B_1 \times \{\,\mathrm{pt}\,\}$. Since $\mathcal{F}$ has rank $n_{\mathcal{G}}$ and $\mathcal{F}$ is transversal to the projection π (since $\mathcal{V}$ is transversal) the image of $\mathcal{F}$ under $T_m\pi$ has rank $n_{\mathcal{G}}$ and must be equal to the tangent space to $B_1 \times \{\,\mathrm{pt}\,\}$. For $\mathcal{G}$ a similar argument works. □

Since the bundles $\mathcal{F}$ and $\mathcal{G}$ project nicely onto B we can lift vectors and vector fields on B to vectors and vector fields on M. Another way of saying this is that $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$ provides a connection for the bundle $M \to B$. We have the following commutative diagram.

$$\begin{tikzcd} M \arrow[d, "\pi"] \\ B \end{tikzcd} \qquad \begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d, "T\pi"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F} \oplus \mathcal{G} \arrow[d, "T\pi_1 \times T\pi_2"] \\ TB \arrow[r] & TB_1 \times TB_2 \end{tikzcd}$$

Remark 10.3.8. The reason for calling the pair $\mathcal{F}$, $\mathcal{G}$ a pair of Darboux integrable distributions is that classical Darboux integrability falls under this definition and one of the main properties is preserved, namely the construction of integral manifold by explicit parameterization.

Suppose we are looking for 2-dimensional integral manifolds of the bundle $\mathcal{V}$ that satisfy the independence condition that at each point the tangent space of the integral manifold intersected with both $\mathcal{F}$ and $\mathcal{G}$ is non-empty. We can parameterize these integral manifolds in the following way. Select a curve γ_1 in B_1 and a curve γ_2 in B_2. The product of these two curves is a surface S in B. The distribution $\mathcal{V}$ restricts on the inverse image of S under π to a Frobenius distribution of rank 2. The leaves of this distribution are integral manifolds of $\mathcal{V}$.⊘

10.3.1 Lie algebras of tangential symmetries

Let $(M, \mathcal{F}, \mathcal{G})$ be a Darboux integrable pair of distributions. Select locally a basis of commuting vector fields $\tilde{F}_1, \ldots, \tilde{F}_{n_\mathcal{G}}$, $n_\mathcal{G} = \operatorname{rank}\mathcal{F}$ for B_1 and $\tilde{G}_1, \ldots, \tilde{G}_{n_\mathcal{F}}$ for B_2, i.e., $[\tilde{F}_i, \tilde{F}_j] = 0$, $[\tilde{G}_i, \tilde{G}_j] = 0$. As vector fields on $B = B_1 \times B_2$ we then automatically have $[\tilde{F}_i, \tilde{G}_j] = 0$. We can lift these vector fields to unique vector fields in M by requiring that the lifted vector fields are contained in $\mathcal{V}$. We write F_i and G_j for the lift of $\tilde{F}_i$ and $\tilde{G}_j$, respectively. Since the vector fields F_i and G_j are contained in $\mathcal{F}$ and $\mathcal{G}$, respectively, their Lie brackets $[F_i, G_j]$ must be contained in $\mathcal{V}$. On the other hand, the projections have Lie bracket $[\tilde{F}_i, \tilde{G}_j]$ equal to zero and therefore $[F_i, G_j]$ must be contained in the tangent space of the fibers of the projection. But $\mathcal{V}$ is transversal to the fibers of the projection and it follows that $[F_i, G_j] = 0$. Since $\mathcal{F}$ has $n_\mathcal{F}$ invariants, the codimension of the completion of $\mathcal{F}$ is equal to $n_\mathcal{F}$ on an open dense subset. The same is true for $\mathcal{G}$. To carry out our constructions we will sometimes need to restrict to a suitable open subset on which this is the case.

We will use the vector fields F_i and G_j to construct various Lie algebras on the fibers of the projection. We only make a choice of vector fields to make the constructions and the proofs easier, most of the Lie algebras that we construct are independent of the choice of F_i and G_j.

We define $\mathfrak{f}$ as the Lie algebra of vector fields (over $\mathbb{R}$, not over $\mathbb{C}^\infty(M)$) generated by F_i, $1 \leq i \leq \operatorname{rank}\mathcal{F}$. This Lie algebra is contained in the Lie algebra of vector fields in the completion of $\mathcal{F}$ and is not necessarily finite-dimensional. We define $\mathfrak{g}$ as the Lie algebra of vector fields generated by G_i, $1 \leq i \leq \operatorname{rank}\mathcal{G}$.

For two vector fields F_i, F_j the Lie bracket $[F_i, F_j]$ is tangent to the fibers of the projection π. This follows from the fact that the F_i are lifts of commuting vector fields and hence $T\pi([F_i, F_j]) = [\tilde{F}_i, \tilde{F}_j] = 0$. This implies that the derived Lie algebras $\mathfrak{f}'$ and $\mathfrak{g}'$ consist of vector fields that are tangential to the projection. Since the generators F_i for $\mathfrak{f}$ are not tangential we find that $\mathfrak{f}' = \mathfrak{f} \cap \mathfrak{z}$. The elements of $\mathfrak{f}'$ commute with the elements in $\mathfrak{g}$ and hence the elements of $\mathfrak{f}'$ are symmetries of $\mathcal{G}$ that are tangential to the projection π.

Definition 10.3.9. Let $(M, \mathcal{F}, \mathcal{G})$ be a Darboux integrable pair of distributions with projection $\pi : M \to B_1 \times B_2$. We define the *tangential symmetries* of $\mathcal{F}$ and $\mathcal{G}$ as the space of all vector fields in $\mathfrak{z}$ that are symmetries of the distributions $\mathcal{F}$ and $\mathcal{G}$, respectively. We write $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$ for the tangential symmetries of $\mathcal{G}$ and $\mathcal{F}$, respectively. ⊖

The name tangential characteristic symmetries was introduced by Vassiliou [66]. The discussion above shows that the vector fields in $\mathfrak{f}'$ are all tangential symmetries of $\mathcal{G}$. We will see below (Lemma 10.3.10) that the tangential symmetries can be expressed in terms of the Lie algebras $\mathfrak{f}'$ and $\mathfrak{g}'$.

For every point $b \in B = B_1 \times B_2$ we write M_b for the fiber $\pi^{-1}(b)$. We write $\mathfrak{z}_b$ for the vector fields on M_b. For the tangential vector fields $\mathfrak{z}$ we define the restriction map

$$\rho_b : \mathfrak{z} \to \mathfrak{z}_b. \tag{10.1}$$

The restriction map is a Lie algebra homomorphism.

Since the vector fields in $\mathfrak{f}'$ and $\mathfrak{g}'$ are tangential we can define the restriction maps $\rho_b : \mathfrak{f}' \to \mathfrak{z}_b$ and $\rho_b : \mathfrak{g}' \to \mathfrak{z}_b$ as well. We denote the image of $\mathfrak{f}'$ under ρ_b by $\mathfrak{f}'_b$ and the image of $\mathfrak{g}'$ under ρ_b by $\mathfrak{g}'_b$. Since the Lie algebras $\mathfrak{f}'$ and $\mathfrak{g}'$ commute, the Lie algebras $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ are commuting Lie algebras of vector fields on M_b.

The distribution $\mathcal{F}^{\text{compl}}$ has codimension $n_{\mathcal{F}}$ on an open dense subset. Therefore on an open dense subset the vector fields in the Lie algebra $\mathfrak{f}'$ span the tangent space to the fiber. We can choose a set of vector fields $X_1, \dots, X_m$ (m is the dimension of the fibers) in $\mathfrak{f}'$ such that the restriction of these vector fields to a fiber M_b is a basis for $\mathfrak{f}'_b$. Let X be a tangential symmetry of $\mathcal{G}$. Since the vector field X is tangential we can write $X = c^j X_j$ for certain functions c^j. Since X is a tangential symmetry, the commutator of X with $\mathcal{G}$ is contained in $\mathcal{G}$ and hence for all $Y \subset \mathcal{G}$

$$[X, Y] \equiv [c^j X_j, Y] \equiv c_j[X_j, Y] - Y(c^j)X_j \equiv Y(c_j)X_j \equiv 0 \mod \mathcal{G}.$$

This implies $Y(c_j) = 0$ for all $Y \subset \mathcal{G}$ and hence the c^j are functions of the invariants of $\mathcal{G}$ only, so $c^j \in \pi_1^*(C^\infty(B_1))$. This proves that the tangential symmetries of $\mathcal{G}$ are a $\pi_1^*(C^\infty(B_1))$-module over the Lie algebra $\mathfrak{f}'$. We have proved

Lemma 10.3.10. *The tangential symmetries of $\mathcal{G}$ are a $\pi_1^*(C^\infty(B_1))$-module over $\mathfrak{f}'$. The tangential symmetries of $\mathcal{F}$ are a $\pi_2^*(C^\infty(B_2))$-module over $\mathfrak{g}'$.*

The Lie algebras $\mathfrak{f}'$ and $\mathfrak{g}'$ depend on the choice of commuting vector fields $\tilde{F}_i$ and $\tilde{G}_j$, respectively. The Lie algebras of tangential symmetries $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$ are independent of this choice. This is clear from Definition 10.3.9. In the lemma below we show that the Lie algebras $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ are also independent of the choice of commuting vector fields.

Lemma 10.3.11. *For every point $b \in B$ the Lie algebras $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ on M_b are invariantly defined reciprocal Lie algebras. The type of the Lie algebra does not depend on the point $b \in B$.*

Proof. The Lie algebra $\mathfrak{f}'$ is tangential to the fibers of the projection. Since $\mathcal{F}$ has only $n_{\mathcal{F}} = \operatorname{rank} \mathcal{G}$ invariants, it follows that for y in an open dense subset of M the image of the evaluation map $\operatorname{ev}(\mathfrak{f}')_y : \mathfrak{f}' \to T_y M_b$ spans the tangent space $T_y M_{\pi(y)}$. This in turn implies that for all points x in an open subset of M_b the evaluation map $\operatorname{ev}(\mathfrak{f}'_b)_x : \mathfrak{f}'_b \to T_x M_b$ is surjective. The same is true for $\mathfrak{g}'_b$. Hence we can apply Lemma 10.1.4 to $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ and conclude that $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ are reciprocal Lie algebras on M_b. From the definitions it follows directly that the Lie algebra $\mathfrak{f}'_b$ only depends on the choice of vector fields F_i, and not on the choice of the vector fields G_j. In the same way $\mathfrak{g}'_b$ does only depend on the choice of G_j. On the other hand, $\mathfrak{f}'_b$ is the centralizer of $\mathfrak{g}'_b$ in the fiber M_b and hence $\mathfrak{f}'_b$ only depends on the choice of G_j. This implies that $\mathfrak{f}'_b$ is invariantly defined.

Make a choice of m vector fields X_i in $\mathfrak{f}'$ such that at each point the vector fields span the tangent space to the fibers of the projection. Locally, near a point $x \in M_b$, we can think of the vector fields X_i as a section of the homomorphism $\rho_b : \mathfrak{f}' \to \mathfrak{f}'_b$.

Since the vector fields X_i span the tangent space to the fiber and the commutator of two tangential vector fields is tangential again, we have $[X_i, X_j] = c^k_{ij} X^k$ for certain functions c^k_{ij}. Since the X_i commute with $\mathfrak{g}$ it follows that the functions c^k_{ij} depend only on the invariants of $\mathcal{G}$. If we restrict to one of the leaves of the completion of $\mathcal{G}$, then the invariants of $\mathcal{G}$ are constant and hence the coefficients c^k_{ij} will be constant. Locally the leaves are foliated by the fibers of the projection and the fact that the coefficients c^k_{ij} depend only on the invariants of $\mathcal{G}$ shows that all fibers M_b in the same leaf of the completion of $\mathcal{G}$ have an isomorphic Lie algebra $\mathfrak{f}'_b$. So if we move in the direction of the completion of $\mathcal{G}$, then the type of $\mathfrak{f}'_b$ does not change. For the same reason the Lie algebras $\mathfrak{g}'_b$ for all fibers M_b in a leaf of the completion of $\mathcal{F}$ have the same type.

The type of $\mathfrak{f}'_b$ is equal to the type of $\mathfrak{g}'_b$ (reciprocal Lie algebras are anti-isomorphic). Therefore if we move in the directions of $\mathcal{F}$ and $\mathcal{G}$ the type of both $\mathfrak{g}'_b$ and $\mathfrak{f}'_b$ does not change. Hence the type of the reciprocal Lie algebras on the fibers is independent of the choice of fiber M_b. $\square$

We conclude that the fibers of the projection carry an invariant structure of two reciprocal Lie algebras. Since the type is locally constant, the type of the Lie algebra is an invariant of the Darboux integrable pair of distributions.

The next step is to extend the Lie algebras on the fibers to Lie algebras on M.

Lemma 10.3.12. *On M there exist finite-dimensional Lie subalgebras L, R of $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$, respectively, such that for all fibers M_b the restriction map ρ_b defines a Lie algebra isomorphism to the Lie algebras on the fibers. The Lie algebras L and R are commuting.*

Proof. Choose a basis of vector fields X_j for $\mathfrak{z}$ contained in $\mathfrak{f}'$. The Lie brackets of the X_j define the structure coefficients

$$[X_i, X_j] = c^k_{ij} X_k. \tag{10.2}$$

We already know that these structure coefficients are functions in $\pi_1^*(C^\infty(B_1))$. Choose a fiber M_{b_0}. Since the type of the Lie algebra is constant for each fiber there exists for every point $b_1 \in B_1$ a linear transformation $\tilde{\mu}(b_1) \in \mathrm{GL}(m, \mathbb{R})$ such that the vector fields $Y_i = \tilde{\mu}(b_1)^j_i X_j$ have the same structure constants as the restrictions of X_j to the fiber M_{b_0}. Locally we can arrange that $\tilde{\mu} : B_1 \to \mathrm{GL}(m, \mathbb{R})$ is a smooth map, see Remark 10.3.13.

The new vector fields Y_j have the structure of a finite-dimensional Lie algebra L and this Lie algebra consists of tangential symmetries of $\mathcal{G}$. In the same way we can construct a finite-dimensional Lie subalgebra R of $\tilde{\mathfrak{g}}$. $\square$

We call L and R *tangential Lie algebras* of symmetries of $\mathcal{G}$ and $\mathcal{F}$, respectively. The choice of a basis for the tangential Lie algebras is not unique. We can for example multiply a basis X_j of L with a matrix $c_j^k \in \mathrm{GL}(m, \mathbb{R}) \otimes \pi_1^*(C^\infty(B_1))$. For all points $x \in M$ the matrix $c_j^k(x)$ acts on the structure constants of L. If for all points x the matrix $c_j^k(x)$ is in the stabilizer of the structure constants, then the vector fields $c_j^k X_k$ define a Lie algebra of tangential symmetries as well.

Remark 10.3.13. Let $\mathcal{L}$ be the space of Lie algebra structures on $\mathbb{R}^m$. This space is an algebraic variety in $C_m = \Lambda^2(\mathbb{R}^m)^* \otimes \mathbb{R}^m$ defined by anti-symmetry and the Jacobi identity. The group $G = \mathrm{GL}(m, \mathbb{R})$ acts on C_m. The vector fields X_j from Lemma 10.3.12 define a map $\mu : B_1 \to C_m$ by assigning to a point $b_1 \in B_1$ the structure constants of X_j in a fiber above b_1 (the structure coefficients are independent of the point in B_2). Since the vector fields X_i are smooth, the map μ is smooth as well. By assumption the image of μ is contained in a single orbit A of the action of G.

Let a be equal to $\mu(x)$. Then the orbit A is equal to G/G_a, where G_a is the stabilizer subgroup of the point a in the orbit. We want to prove that there is a smooth lift of the map μ to a map $\tilde{\mu} : B_1 \to \mathrm{GL}(m, \mathbb{R})$ such that the diagram below is commutative.

$$\begin{tikzcd} & \mathrm{GL}(m) \arrow[d] \\ B_1 \arrow[ur, dashed, "\tilde{\mu}"] \arrow[r, "\mu"] & A \end{tikzcd}$$

We have to be carefull here because the map μ is continuous with respect to the topology on A induced from the surrounding space C_m. The structure on A as the homogeneous space G/G_a might be different. From the theory in Section A.3 it follows that μ is a smooth map to G/G_a. The projection $G \to G/G_a$ is a principal fiber bundle and hence there is a smooth lift of μ. ⊘

Theorem 10.3.14. *Let $(M, \mathcal{F}, \mathcal{G})$ be a Darboux integrable system. Let $n_{\mathcal{F}}$ and $n_{\mathcal{G}}$ be the rank of $\mathcal{F}$ and $\mathcal{G}$, respectively. Then locally there is a unique local Lie group H such that the manifold M is of the form*

$$B_1 \times B_2 \times H,$$

with $B_1 \subset \mathbb{R}^{n_{\mathcal{G}}}$, $B_2 \subset \mathbb{R}^{n_{\mathcal{F}}}$. The Darboux projection π is given by the projection on $B_1 \times B_2$. The tangential symmetries of $\mathcal{F}$ and $\mathcal{G}$ are tangent to the fibers of π and restricted to each fiber they form reciprocal Lie algebras. The left- and right-invariant vector fields on H define the tangential Lie algebras of symmetries on M.

Example 10.3.15 (Wave equation). On $\mathbb{R}^5$ with coordinates x, y, z, p, q introduce the following two distributions:

$$\mathcal{F} = \mathrm{span}\left(\partial_x + p\partial_z, \partial_p\right), \quad \mathcal{G} = \mathrm{span}\left(\partial_y + q\partial_z, \partial_q\right).$$

The invariants of $\mathcal{F}$ are y, q and the invariants of $\mathcal{G}$ are x, p. The Darboux projection is given by

$$\pi : M \to \mathbb{R}^2 \times \mathbb{R}^2 : (x, y, z, p, q) \mapsto (x, p) \times (y, q).$$

As a set of commuting vector fields we take $\tilde{F}_1 = \partial_x$, $\tilde{F}_2 = \partial_p$, $\tilde{G}_1 = \partial_y$, $\tilde{G}_2 = \partial_y$. The lifts to vector fields on M are given by

$$\begin{aligned} F_1 &= \partial_x + p\partial_z, \quad & F_2 &= \partial_p, \\ G_1 &= \partial_y + q\partial_z, \quad & G_2 &= \partial_q. \end{aligned}$$

Define $F_3 = [F_1, F_2] = -\partial_z$ and $G_3 = [G_1, G_2] = -\partial_z$. The fibers of the projection are isomorphic to $\mathbb{R}$ with coordinate z. On the leaves of the completion of $\mathcal{G}$ (i.e., x and p constant) we have a 3-dimensional Lie algebra with structure equations

$$[F_1, F_2] = F_3, \quad [F_1, F_3] = 0, \quad [F_2, F_3] = 0.$$

The same holds for the leaves of the completion of $\mathcal{F}$. The fibers of the projection are 1-dimensional and have the structure of a 1-dimensional abelian Lie algebra (generated by the vector field ∂_z). ⊘

In each fiber M_b of the projection, we have the center of the Lie algebra on the fiber. The center of $\mathfrak{f}'_b$ is equal to the center of $\mathfrak{g}'_b$. We write $\mathfrak{c}_b$ for the center of the Lie algebra on M_b. The distribution $\mathcal{C}$ spanned by $\mathfrak{c}_b$, $b \in B$, is integrable and hence defines a local foliation of the fibers.

Lemma 10.3.16. *The tangential vector fields that are symmetries of both $\mathcal{F}$ and $\mathcal{G}$ are the tangential vector fields for which the restriction to a fiber is in the center of the Lie algebra of tangential symmetries. In particular, these vector fields are contained in $\mathcal{C}$.*

Proof. Restricted to each fiber M_b the vector field is invariant under both $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$. By Theorem 10.2.2 the restriction must be contained in $\mathfrak{c}_b$. □

This lemma shows that most Darboux projections are not a symmetry reduction. The Darboux projection can be a symmetry reduction if and only if the Lie group associated to the equation is abelian. All Darboux integrable equations or systems with non-abelian Lie group are examples of true vector pseudosymmetries.

Remark 10.3.17. Given a Lie group there is no systematic way of constructing a corresponding Darboux integrable system. Even if we can construct such a system, there is no guarantee that this system corresponds to a prolonged first order system or second order equation. Despite this, Vessiot [69, 70] succeeded in using the classification of 3-dimensional Lie algebras to construct a classification of all hyperbolic Goursat equations. ⊘

Example 10.3.18. Consider the second order equation

$$s = \frac{az}{(x+y)^2}. \tag{10.3}$$

This equation is Darboux integrable on the $k+1$-jets if $a = k(k+1)$. Hence for $a = k(k+1)$ the $(k-1)$-th prolonged equation manifold has dimension $5+2k$ and the prolonged Monge systems define a Darboux integrable pair of distributions.

We work out the case $k = 2$ in detail. The prolonged equation manifold has coordinates $x, y, z, p, q, r, t, z_{xxx}, z_{yyy}$. The prolonged characteristic systems are $\mathcal{F} = \mathrm{span}(F_1, F_2)$, $\mathcal{G} = \mathrm{span}(G_1, G_2)$ with

$$\begin{aligned}
F_1 &= \partial_x + p\partial_z + r\partial_p + \sigma\partial_q + z_{xxx}\partial_r + Y(\sigma)\partial_t + Y(Y(\sigma))\partial_{z_{yyy}},\\
F_2 &= \partial_{z_{xxx}},\\
G_1 &= \partial_y + q\partial_z + \sigma\partial_p + t\partial_q + X(\sigma)\partial_r + z_{yyy}\partial_t + X(X(\sigma))\partial_{z_{xxx}},\\
G_2 &= \partial_{z_{yyy}}.
\end{aligned}$$

Here $\sigma = 6z/(x+y)^2$, $X = \partial_x + p\partial_z + r\partial_p + \sigma\partial_q$ and $Y = \partial_y + q\partial_z + \sigma\partial_p + t\partial_q$. Both characteristic systems have two invariants,

$$\begin{aligned}
I_{\mathcal{F}} &= \left\{ y,\, z_{yyy} + 6\frac{q + t(x+y)}{(x+y)^2} \right\}_{\mathrm{func}},\\
I_{\mathcal{G}} &= \left\{ x,\, z_{xxx} + 6\frac{p + r(x+y)}{(x+y)^2} \right\}_{\mathrm{func}}.
\end{aligned}$$

If we make a transformation to the variables $\tilde{x} = x$, $\tilde{y} = y$, $\tilde{z} = z$, $\tilde{p} = p$, $\tilde{q} = q$, $\tilde{r} = r$, $\tilde{t} = t$, $\tilde{a} = z_{xxx} + 6(p + r(x+y))/(x+y)^2$, $\tilde{b} = z_{yyy} + 6(q + t(x+y))/(x+y)^2$, then we can easily construct the Darboux projection, choose commuting vector fields, lifts these vector fields to M and calculate the tangential symmetries. Translated back to the original variables we find that the tangential symmetries of $\mathcal{F}$ are spanned by

$$\begin{aligned}
L_1 &= \partial_t - 6H\partial_{z_{yyy}},\\
L_2 &= \partial_q - 6H\partial_t + 30H^2\partial_{z_{yyy}},\\
L_3 &= \partial_z - 6H\partial_q + 24H^2\partial_b - 108H^3\partial_{z_{yyy}},\\
L_4 &= H\partial_z - H^2\partial_p - 3H^2\partial_q + 2H^3\partial_r\\
&\quad + 10H^3\partial_t + 6H^4\partial_{z_{xxx}} + 42H^4\partial_{z_{yyy}},\\
L_5 &= H^2\partial_z - 2H^3(\partial_p + \partial_q)\\
&\quad + 6H^4(\partial_r + \partial_t) - 24H^4(\partial_{z_{xxx}} + \partial_{z_{yyy}}),
\end{aligned}$$

with $H = (x+y)^{-1}$. The tangential symmetries of $\mathcal{G}$ are given by the same expressions, but with x, p, r and z_{xxx} replaced by y, q, t and z_{yyy}, respectively. The tangential symmetries commute and the Lie group associated to this Darboux integrable system is the 5-dimensional abelian Lie group. $\oslash$

10.3.2 First order Darboux integrable systems

In Section 8.2.2 we made a complete classification of the (2, 2)-Darboux integrable hyperbolic first order systems. We found that under contact transformations there are only two equivalence classes: the almost product system and the affine system. In the examples below we will calculate the tangential symmetries for these systems.

In Vassiliou [65, Theorem 3] a normal form for Darboux integrable hyperbolic first order systems is given. Vassiliou uses the low dimensions of the distributions $\mathcal{F}$, $\mathcal{G}$ and the manifold M to arrive at a normal form in local coordinates for the lifts of the commuting vector fields and the tangential Lie algebras. For this class of equations his normal form is a much stronger form than our constructions (which are valid in a more general setting). His normal form allows Vassiliou to conclude that the type of the Lie algebra is locally a full invariant for the hyperbolic Darboux integrable first order systems. In [65, Theorem 5] Vassiliou gives examples of almost product and hyperbolic affine Darboux integrable systems.

Example 10.3.19 (Tangential symmetries for the hyperbolic affine Darboux integrable system). We will discuss the tangential symmetries of the system $\partial w/\partial z^F = ww^F$. An adapted coframing for this system was given in (8.10). In the coordinates z, w, p the dual framing is given by

$$\begin{aligned} \partial_\theta &= w\partial_w, \\ \partial_\omega &= w^{-1}\big(\partial_z + w(p+w)\partial_w + ww^F\partial_{w^F}\big), \\ \partial_\pi &= w\partial_p. \end{aligned} \tag{10.4}$$

The characteristic systems are given by the components of $\operatorname{span}(\partial_\omega, \partial_\pi)$. The Darboux projection is onto the variables z, p. The vector fields tangent to the Darboux projection are spanned by the components of $\partial_\theta = w\partial_w$. The tangential symmetries are given by

$$X_1 = \partial_{\theta^F} = w^F\partial_{w^F}, \quad X_2 = w^F\partial_\theta + w^F\partial_{\theta^F}.$$

The commutator is $[X_1, X_2] = X_2$. The components $(X_1)^1$, $(X_2)^1$ span the tangential Lie algebra of symmetries of $\mathcal{F}$, the components $(X_1)^2$, $(X_2)^2$ span the tangential Lie algebra of symmetries of $\mathcal{G}$.

If we write $z = (z_1, z_2)^T$, $w = (w_1, w_2)^T$ and $p = (p_1, p_2)^T$, then the characteristic systems are

$$\begin{aligned} \mathcal{F} &= \operatorname{span}\Big((w_1)^{-1}\left(\partial_{z_1} + w_1(p_1+w_1)\partial_{w_1} + w_1w_2\partial_{w_2}\right), w_1\partial_{p_1}\Big), \\ \mathcal{G} &= \operatorname{span}\Big((w_2)^{-1}\left(\partial_{z_2} + w_2(p_2+w_2)\partial_{w_2} + w_2w_1\partial_{w_1}\right), w_2\partial_{p_2}\Big). \end{aligned}$$

The tangential symmetries of $\mathcal{F}$ are given by $(X_1)^1 = w_2\partial_{w_2}$ and $(X_2)^1 = w_2(w_1\partial_{w_1} + w_2\partial_{w_2})$. ⊘

Example 10.3.20 (Tangential symmetries for the almost product Darboux integrable system). We consider the equation $\partial w / \partial z^F = w^F/(1 - z z^F)$. An adapted coframing in the coordinates z, w, p is given in (8.19). We have $\partial_\theta = \partial_w$ and the characteristic systems are given by the components of

$$\mathcal{F} = \operatorname{span}\left(\partial_z + \left(p + \frac{w z^F}{1 - z z^F}\right)\partial_w + \left(\frac{w^F}{1 - z z^F}\right)^F \partial_{w^F}, \partial_p \right) \subset \mathbb{D} \otimes \mathcal{V}.$$

Let $H = 1 - z z^F$. The tangential Lie algebras of symmetries are given by the components of

$$X_1 = \partial_{w^F}, \quad X_2 = \frac{1}{H}\partial_w + \frac{1}{z^F H}\partial_{w^F},$$

with $[X_1, X_2] = 0$. $\oslash$

Chapter 11

Projections

In the previous chapters we have seen special cases of a more general method to analyze systems of partial differential equations. The basic idea is the following. Suppose we have some geometric structure on a manifold M. Given a projection $\pi : M \to B$ to some lower dimensional manifold, we can try to carry the structure on M over to a structure on B. This is not always possible and often the structure on B will contain less information than the original structure on M. But sometimes the remaining structure on B is enough, for example to find solutions to a partial differential equation. We call all these methods "projection" methods. In this chapter we will give a brief overview of the different projection methods and give more examples.

In this dissertation the concept of a (vector) pseudosymmetry has been used to give a unified description of various existing methods (symmetry reductions, Darboux integrability, pseudoholomorphic curves) and some new methods (base projections, true pseudosymmetries) related to partial differential equations. We hope the reader has seen that these pseudosymmetries can be useful in the context of partial differential equations. But even in other parts of mathematics the concept might be useful. Whenever a symmetry is used in a mathematical construction, it is possible that a pseudosymmetry can be used in the same construction for a wider class of objects. Hopefully, in the future pseudosymmetries can be found and applied to more fields in mathematics.

11.1 Projection methods

In this section we review the different projection methods. In Figure 11.1 there is a schematic picture of the different methods. All projections methods described have in common that they can be reduced to the projection of certain distributions or structures constructed from the distributions (such as almost product structures, Monge systems, etc.) on a manifold to the quotient manifold. An important difference between many projections is that some of the projections are transversal to the distributions (generalized Darboux projections) and some of the projections have fibers contained in the distributions (the base projections in Chapter 7).

Symmetry reductions. Symmetry methods for partial differential equations were introduced by Sophus Lie in the 18th century. The literature on this subject and the many applications are enormous. For some basic introductions and more references see Olver [58], Duzhin and Chebotarevsky [30] and Hydon [42].

For any system of ordinary differential equations or partial differential equations with a symmetry we can take the quotient of the system by this symmetry. On the quotient manifold there will be a new system of equations. Two examples of symmetry reductions of ordinary differential equations are given below.

Example 11.1.1. Consider the second order ordinary differential equation

$$z'' = (y^2 + x^{-2})^{3/2}.$$

For the equation manifold we can use coordinates $x, z, p = \partial z/\partial x$. The contact distribution is given by $\mathrm{span}(\partial_x + p\partial_z + (y^2 + x^{-2})^{3/2}\partial_p)$. A symmetry of the contact distribution is given by the vector field $V = -x\partial_x + z\partial_z + 2p\partial_p$. Two invariants of the vector field are $X = xz$, $P = x^2 p$. If we make a projection $(x, z, p) \mapsto (X, P)$, then the contact distribution is projected to the rank 1 distribution $\mathrm{span}(\partial_X + (2P + (X^2+1)^{3/2})/(P + X)\partial_P)$. Hence the equation is reduced to the first order ordinary differential equation

$$P' = \frac{2P + (X^2+1)^{3/2}}{P + X}. \qquad \oslash$$

Example 11.1.2. Consider the ordinary differential equation $y' = (2/5)(y^2 + x^{-2})$. The equation has a one-parameter symmetry group generated by $-x\partial_x + y\partial_y$. The finite version of the symmetry is $(x, y) \mapsto (\exp(-\epsilon)x, \exp(\epsilon)y)$. We can solve this equation by introducing new coordinates $\tilde{x}, \tilde{y}$ in which the symmetry takes the simple form $\partial_{\tilde{y}}$. We take $\tilde{x} = xy$, $\tilde{y} = \ln(y)$. The symmetry V is given in the new coordinates by $\partial_{\tilde{y}}$ and the equation transforms to

$$\tilde{y}' = \frac{2(\tilde{x}^2 + 1)}{\tilde{x}(2\tilde{x}^2 + 5\tilde{x} + 2)}.$$

The new equation does not depend on $\tilde{y}$ and can be integrated at once. The general solution is

$$\tilde{y} = \frac{5}{3}\ln(2\tilde{x} + 1) - \ln(\tilde{x}) - \frac{5}{3}\ln(\tilde{x} + 2) + C_1,$$

with C_1 an arbitrary integration constant. Transforming back to the original variables and solving for y yields the solution

$$y = \frac{3}{2(x - Cx^{8/5})} - \frac{2}{x}. \qquad \oslash$$

Base projections. Base projections are projections for which the tangent spaces to the fibers are contained in the contact distribution. The projections of first order systems to a base manifold with almost product or almost complex structure and the projections of Monge-Ampère equations to the first order contact bundle are examples of base projections.

For example for a Monge-Ampère equation $(M, \mathcal{V})$ the fibers of the projection are generated by the vector fields in $C(\mathcal{V}')$ and these are contained in the contact distribution $\mathcal{V}$.

Darboux integrability. The method of Darboux can be formulated as a projection method for hyperbolic (or elliptic) exterior differential systems of class $s > 0$. For Darboux integrability we need enough functionally independent invariants such that projection onto the space of invariants defines a transversal projection.

The Darboux projections have the special property that the projected structure is integrable. This means that we can explicitly give all solutions of the projected structure in terms of holomorphic functions (elliptic systems) or two functions of one variable (hyperbolic systems). These solutions can then be lifted to solutions of the original system by an integration procedure.

We mention here that most Darboux integrable equations are not generated by symmetries of the equation, so in general the method does not fall under the symmetry reductions. In Chapter 10 we showed that a Darboux projection is generated by symmetries if and only if the Lie group associated to the equation is abelian.

Pseudosymmetries. We have described pseudosymmetries as the projections from second order equations to first order systems, the projections of first order systems to Monge-Ampère equations and the projections of Monge-Ampère equations to almost product or almost complex structures. See sections 9.2.2–9.2.4. The calculation of pseudosymmetries is be rather difficult since the equations are non-linear equations. In contrast, the equations for symmetries of a system of partial differential equations are linear equations. Nevertheless, for second order scalar equations we could use the geometry of the system to simplify the calculation of pseudosymmetries.

Vector pseudosymmetries. All examples of projection methods mentioned above fall under the general definition of a vector pseudosymmetry. Typical examples of vector pseudosymmetries are second order equations with a 3-dimensional symmetry group of translations or first order systems with a 2-dimensional symmetry group of translations. Examples are the wave equation, the Laplace equation and the minimal surface equation.

Also the Darboux integrable equations are examples of vector pseudosymmetries. In the next section there are two examples (Example 11.2.3 and Example 11.2.4) of systems with vector pseudosymmetries that are neither symmetry reductions nor Darboux projections. In the literature the author has not found examples of the use of vector pseudosymmetries for systems that are not Darboux integrable and do not have enough symmetries to make a projection.

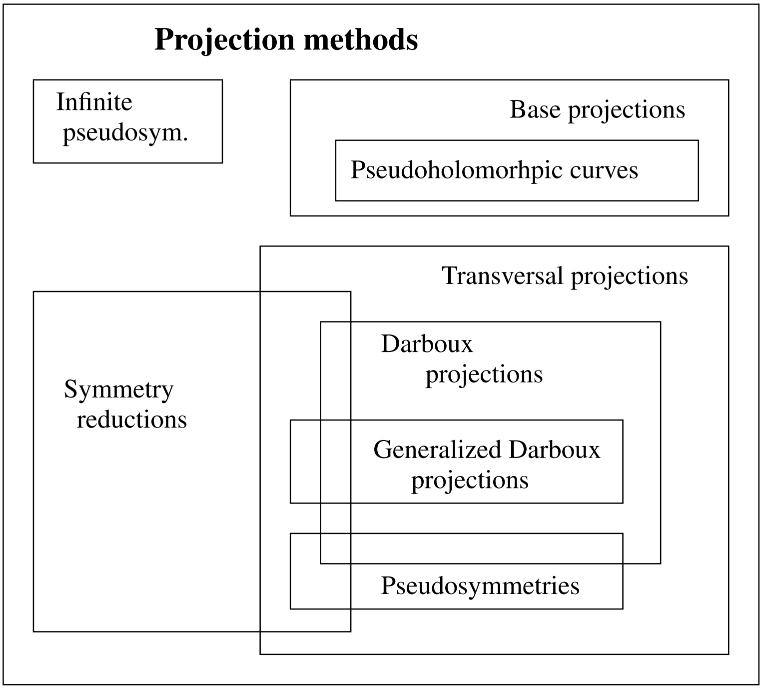

Figure 11.1: Overview of the different projection methods

Infinite pseudosymmetries. The pseudosymmetries on infinite jet bundles have been described in a couple of examples in Section 9.5.2. In this dissertation we did not develop the machinery to work on these manifolds, but we hope the examples have made clear that some of the classic Bäcklund transformations can be understood using pseudosymmetries. In Section 9.5.1 also the relation to integral extensions is explained.

11.2 Examples

Example 11.2.1 (Telegraph equation). The telegraph equation $s = z$ has characteristic systems

$$\mathcal{F} = \operatorname{span}\left(\partial_x + p\partial_z + r\partial_p + z\partial_q + q\partial_t, \partial_r\right),$$
$$\mathcal{G} = \operatorname{span}\left(\partial_y + q\partial_z + z\partial_p + t\partial_q + p\partial_r, \partial_t\right).$$

The invariants of the systems are $I_{\mathcal{F}} = \{y\}_{\text{func}}$ and $I_{\mathcal{G}} = \{x\}_{\text{func}}$. With the method of Laplace [44] we can prove this equation is not Darboux integrable at any order.

The symmetries ∂_x, ∂_y and $V = z\partial_z + p\partial_p + q\partial_q + r\partial_r + t\partial_t$ span a rank 3 integrable distribution that is transversal to $\mathcal{V}$ at generic points. Hence the projection onto the invariants of this distribution defines a generalized Darboux projection. For the generalized Darboux projection we can use the projection variables $\tilde{p} = p/z, \tilde{q} = q/z, \tilde{r} = r/z, \tilde{t} = t/z$. The projected systems are

$$\tilde{\mathcal{F}} = \operatorname{span}\big((-\tilde{p}^2 + \tilde{r})\partial_{\tilde{p}} + (1 - \tilde{p}\tilde{q})\partial_{\tilde{q}} - \tilde{p}\tilde{t}\partial_{\tilde{t}}, \partial_{\tilde{r}}\big),$$
$$\tilde{\mathcal{G}} = \operatorname{span}\big((-\tilde{q}^2 + \tilde{t})\partial_{\tilde{q}} + (-1 - \tilde{p}\tilde{q})\partial_{\tilde{p}} - \tilde{q}\tilde{r}\partial_{\tilde{r}}, \partial_{\tilde{t}}\big).$$

The projected system defines an almost product structure on the manifold with coordinates $\tilde{p} = p/z, \tilde{q} = q/z, \tilde{r} = r/z, \tilde{t} = t/z$. ⊘

Example 11.2.2 (Liouville equation). The Liouville equation

$$s = \exp z \tag{11.1}$$

is a very well-known equation. The equation has characteristic subsystems

$$\mathcal{F} = \operatorname{span}\left(\partial_x + p\partial_z + r\partial_p + \exp(z)\partial_q + q\exp(z)\partial_t, \partial_r\right),$$
$$\mathcal{G} = \operatorname{span}\left(\partial_y + q\partial_z + \exp(z)\partial_p + +t\partial_q + p\exp(z)\partial_r, \partial_t\right).$$

The invariants for the two Monge systems are $I_{\mathcal{F}} = \{y, t - q^2/2\}_{\mathrm{func}}$ and $I_{\mathcal{G}} = \{x, r - p^2/2\}_{\mathrm{func}}$. Since each system has two invariants, the equation is Darboux integrable. The Darboux projection is generated by the vector fields ∂_z, $\partial_p + p\partial_r$ and $\partial_q + q\partial_t$.

The tangential characteristic symmetries of $\mathcal{F}$ are

$$R_1 = \partial_q + q\partial_t, \quad R_2 = \partial_z + q\partial_q + q^2\partial_t,$$
$$R_3 = q\partial_z + \exp(z)\partial_p + (q^2/2)\partial_q + \exp(z)p\partial_r + (q^3/2)\partial_t,$$

and the tangential characteristic symmetries of $\mathcal{G}$ are

$$L_1 = \partial_p + p\partial_t, \quad L_2 = \partial_z + p\partial_p + p^2\partial_r,$$
$$L_3 = p\partial_z + (p^2/2)\partial_p + \exp(z)\partial_q + (p^3/2)\partial_r + \exp(z)q\partial_t.$$

The Lie algebras generated by L_j and R_j are reciprocal and isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

A calculation of the symmetries of the equation reveals that all point symmetries are prolongations of $\xi\partial_x + \eta\partial_y + (\xi_x + \eta_y)\partial_z$ for arbitrary functions ξ, η. A symmetry of the equation is $V = x\partial_x - \partial_z - p\partial_p - 2r\partial_r$. For $x \neq 0$ the projection $\pi(m) = (x\exp z, y, px, q, rx^2, t)$ projects the Liouville equation to a first order system. ⊘

Example 11.2.3 (Non-symmetry projection). So far we have seen only projections that were either generated by symmetries of the equation or Darboux projections, i.e., generated by invariants. There are also projections that are not generated by these symmetries and are also not a Darboux projection.

Consider the class of hyperbolic second order equations $r - g(x, y)^2 t = h(x, y)$ for arbitrary functions g, h with $g(0) = 1$, $g'(0) \neq 0$. For general g, h this equation is not Darboux integrable and has small symmetry group; the characteristic systems are

$$\mathcal{F} = \operatorname{span}\left(D_x - g(x, y) D_y + \left(2g(x, y)t \frac{\partial g(x, y)}{\partial y} + \frac{\partial h(x, y)}{\partial y} \right) \partial_s, \partial_t + g(x, y) \partial_s \right),$$
$$\mathcal{G} = \operatorname{span}\left(D_x + g(x, y) D_y + \left(2g(x, y)t \frac{\partial g(x, y)}{\partial y} + \frac{\partial h(x, y)}{\partial y} \right) \partial_s, \partial_t - g(x, y) \partial_s \right),$$

with $D_x = \partial_x + p\partial_z + (g(x, y)^2 t + h(x, y))\partial_p + s\partial_q$, $D_y = \partial_y + q\partial_z + s\partial_p + t\partial_q$. From the form of the characteristic systems above, we can easily see that the distributions $\operatorname{span}(\partial_z)$ and $\operatorname{span}(\partial_z, \partial_p, \partial_q)$ generate a projection π_1 to a first order system and a transversal projection π_3 to a four-dimensional almost product manifold, respectively. For special equations, e.g., $g(x, y) = 1 + x^2$, $h(x, y) = 0$ or $g(x, y) = 1 + \exp x$, $h(x, y) = 0$ one can indeed check that the equation is not Darboux integrable and the projection π_3 is not a symmetry projection.

The projection π_3 has some another interesting features. If we take the quotient of the system by the symmetry ∂_z we find a first order system equivalent to $u_x - g(x, y)^2 v_y = h(x, y)$, $u_y = v_x$. The vector fields ∂_p and ∂_q are no symmetries of the original equation, but the projections to the first order systems are symmetries. So while the projection π_3 is not a symmetry projection it factorizes through 2 symmetry projections.

Note that even though the Nijenhuis tensor restricted to $\mathcal{V}' \times_M \mathcal{V}'$ is identically zero, the projected structure is not integrable. This is because the projection does respect the structure on $\mathcal{V}$, but does not respect the structure on $\mathcal{V}'$. ⊘

Example 11.2.4. Consider the class of hyperbolic equations $r - h(t) = \phi(z)$ for a arbitrary functions h, ϕ with $h(0) = 1$, $h'(0) \neq 0$. For generic h, ϕ this equation is not Darboux integrable and has symmetry group too small to make transversal projections. The characteristic systems are

$$\mathcal{F} = \operatorname{span}\big(D_x + \sqrt{h'(t)} D_y + q\phi'(z)\partial_s, \partial_t + \sqrt{h'(t)}\partial_s\big),$$
$$\mathcal{G} = \operatorname{span}\big(D_x - \sqrt{h'(t)} D_y + q\phi'(z)\partial_s, \partial_t - \sqrt{h'(t)}\partial_s\big).$$

We see the equation always has a transversal projection onto the variables x, y, t, s. The equation also has projections to a first order system generated by either one of the symmetries ∂_x, ∂_y. ⊘

Example 11.2.5 ($r + t = p + q$). In the elliptic case it is often more convenient to work with the complex structure on the bundle $\mathcal{V}$, than complexifying the system (which can only be done in the analytic setting), applying the hyperbolic theory and converting the results back. The equation $r + t = p + q$ is elliptic and analytic. We use the equation to eliminate the variable r. The complex characteristic systems on the equation manifold M are

$$\mathcal{F} = \operatorname{span}\left(D_x + iD_y + (s + t)\partial_t, \partial_s - i\partial_t\right),$$
$$\mathcal{G} = \operatorname{span}\left(D_x - iD_y + (s + t)\partial_t, \partial_s + i\partial_t\right),$$

with $D_x = \partial_x + p\partial_z + (p+q-t)\partial_p + s\partial_q$, $D_y = \partial_y + q\partial_z + s\partial_p + t\partial_q$. The invariants are $I_{\mathcal{F}} = \{x+iy\}_{\text{func}}$, $I_{\mathcal{G}} = \{x-iy\}_{\text{func}}$. The equation has only one invariant for each of the characteristic systems and is therefore not Darboux integrable on the second order equation manifold.

- The equation is quasi-linear. Hence the projection $M \to B : m \mapsto (x, y, u, v)$ is a base projection that intertwines the almost complex structure on M with an almost complex structure on the base manifold B. The first order system M is equivalent to the system of equations for pseudoholomorphic curves in B.

- The equation is translation invariant and the infinitesimal symmetry $V = \partial_z$ generates the projection $\pi : M \to B : m \mapsto (x, y, p, q, r, s)$. The projected manifold has the structure of an elliptic first order system that is equivalent to the system $u_y = v_x$, $u_x = -v_y + u + v$.

- The equation is translation invariant and the vector fields ∂_x, ∂_y, ∂_z generate a transversal projection $\pi : M \to B : m \mapsto (p, q, r, s)$. The characteristic systems are projected to the distributions

  $$\tilde{\mathcal{F}} = \operatorname{span}(e_1 + ie_2, e_3 + ie_4), \quad \tilde{\mathcal{G}} = \operatorname{span}(e_1 - ie_2, e_3 - ie_4),$$

  with

  $$\begin{aligned} &e_1 = (-t+p+q)\partial_p + s\partial_q, \quad e_2 = s\partial_p + t\partial_q + (s+t)\partial_t, \\ &e_3 = \partial_s, \quad e_4 = -\partial_t. \end{aligned}$$

  The projection maps $\mathcal{V}$ onto TB at the points $(p+q-t)t - s^2 \neq 0$. At such points the complex structure on $\mathcal{V}$ is pushed down to an almost complex structure on B. With respect to the basis e_1, e_2, e_3, e_4 this almost complex structure is given by the matrix

  $$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{11.2}$$

  The structure is not integrable as can be checked by calculating the corresponding Nijenhuis tensor.

- A second transversal projection is provided by the map $\pi : M \to B : m \mapsto (x, y, r, s)$. In this case we have $e_1 = \partial_x$, $e_2 = \partial_y + (s+t)\partial_t$, $e_3 = \partial_s$, $e_4 = -\partial_t$ and the projected almost complex structure with respect to this basis e_j again has the form (11.2).

  A 2-plane given by $\operatorname{span}(\partial_x + \alpha\partial_s + \beta\partial_t, \partial_y + \gamma\partial_s + \delta\partial_t)$ is complex with respect to the almost complex structure if and only if $\beta + \gamma = 0$ and $\delta - \alpha - (s+t) = 0$. The equations for pseudoholomorphic curves for the almost complex structure on the projected manifold therefore correspond to the first order elliptic system

  $$s_y + t_x = 0, \quad t_y - s_x = s + t. \qquad \oslash$$

Example 11.2.6 (continuation of Example 8.1.6). In Example 8.1.6 we found that the linear elliptic first order system defined by

$$\frac{\partial w}{\partial \bar{z}} = b(z, \bar{z})\bar{w}$$

is Darboux integrable for certain functions b. In this example we will construct projections from the equation manifold of this first order system to a 4-dimensional base manifold with an integrable almost complex structure such that the projection intertwines the complex structure on the contact distribution with the complex structure on the base manifold.

We will start with a description of the Darboux projection in the case the system is Darboux integrable. Then we will consider another projection which is different from the Darboux projection.

Darboux projection. Inspired by the Darboux integrability we consider a projection onto the pair of complex variables $\tilde{z}$, $\tilde{p}$. One can easily see that the contact distribution dual to θ is mapped by the tangent map of the projection onto the tangent space of the projected space, i.e., the projection is transversal to $\mathcal{V} = \theta^{\perp}$. To see that the complex structure on $\mathcal{V}$ is also preserved we first apply the coordinate transformation $\tilde{z} = z$, $\tilde{w} = w$, $\tilde{p} = p - \partial \log(b)/\partial z$.

The adapted coframing (8.1) is given in the new coordinates by

$$\begin{aligned}
\tilde{\theta} &= \mathrm{d}\tilde{w} - (\tilde{p} + (b_{\bar{z}}/b)\tilde{w})\mathrm{d}z - b\bar{w}\mathrm{d}\bar{z},\\
\tilde{\omega} &= \mathrm{d}\tilde{z},\\
\tilde{\pi} &= \mathrm{d}\tilde{p} + \mathrm{d}((b_{\bar{z}}/b)\tilde{w}) - (b_{\bar{z}}\bar{w} + b\bar{q})d\bar{z}\\
&= \mathrm{d}\tilde{p} + (b_z/b)\theta + (b_{\bar{z}}/b)(\tilde{p} + (b_{\bar{z}}/b)w)\omega + (b_z/b)b\bar{w}\bar{\tilde{\omega}}\\
&\quad + \Big(\frac{\partial^2 \log b}{\partial \tilde{z}^2}\Big)\tilde{w}\omega + |b|^2\tilde{w}\bar{\omega} - (b_{\bar{z}}\bar{w} + |b|^2\tilde{w})\bar{\tilde{\omega}}\\
&= \mathrm{d}\tilde{p} + (b_z/b)\theta + (b_{\bar{z}}/b)(\tilde{p} + (b_{\bar{z}}/b)w)\omega\\
&\quad + (b_z/b)b\bar{w}\bar{\tilde{\omega}} + \Big(\frac{\partial^2 \log b}{\partial \tilde{z}^2}\Big)\tilde{w}\omega - b_{\bar{z}}\bar{w}\bar{\tilde{\omega}}.
\end{aligned}$$

If we want we can adapt the coframing such that $\tilde{\pi} = d\tilde{p}$. The structure equations for $\tilde{\theta}$ are

$$\tilde{\theta} \equiv -\tilde{\pi} \wedge \tilde{\omega} \mod \tilde{\theta}, \bar{\tilde{\theta}}.$$

A complex basis for the complexified contact distribution $\mathcal{V} \otimes \mathbb{C}$ is given in these coordinates by the vector fields $X = \partial_{\tilde{z}} + (\tilde{p} + (b_z/b)w)\partial_{\tilde{w}} + b\bar{\tilde{w}}\partial_{\bar{\tilde{\omega}}}$, $\bar{X}$, $P = \partial_{\tilde{p}}$ and $\bar{P}$. Since these vector fields are dual to the adapted coframing, the complex structure with respect to the basis of $\mathcal{V}$ defined by the vector fields is constant. The projection onto the coordinates $\tilde{z}$, $\tilde{p}$ intertwines the complex structures.

Non-Darboux projection. We use the coordinates $x, y, u, v, P = u_x, Q = u_y$ on the equation manifold. We will use real coordinates from now on. We use a capital P and Q here to avoid confusion with the p and q introduced before. A basis for the contact distribution in real coordinates is given by

$$\begin{aligned}
&\partial_x + P\partial_u + (-Q + \beta v + \alpha u)\partial_v,\\
&\partial_y + Q\partial_u + (P + \beta u - \alpha v)\partial_v,\\
&\partial_P,\ \partial_Q.
\end{aligned}$$

Here we have written $b(z, \bar{z}) = \alpha(x, y) + i\beta(x, y)$. The complex structure on the distribution with respect to this basis is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & G & 0 & -1\\ -G & 0 & 1 & 0 \end{pmatrix},$$

with $G = \beta_y v + \beta^2 u + \alpha_y u + \alpha_x v + \alpha^2 u - \beta_x u$. The condition that the projection onto the variables x, y, P, Q is a good projection, is precisely that the matrix J is independent of the coordinates u and v, i.e.,

$$\alpha_y - \beta_x + \alpha^2 + \beta^2 = 0, \quad \alpha_x + \beta_y = 0. \tag{11.3}$$

If the two conditions (11.3) are satisfied, then we have a good projection and moreover, G is identically zero so the projected almost complex structure is integrable. A calculation yields that these two conditions (11.3) imply that $\partial^2(\log b)/\partial z\partial\bar{z} = |b|^2/4$ (notice the factor 4), although the converse is not true. So unless $b = 0$ the Darboux integrable systems do not have a projection onto the variables x, y, P, Q. ⊘

Example 11.2.7. Every non-generic affine Darboux integrable hyperbolic first order system can be written in local coordinates as

$$u_y = uv, \quad v_x = vu.$$

An adapted coframing is given by

$$\begin{aligned}
\theta^1 &= \mathrm{d}u - p\mathrm{d}x - uv\mathrm{d}y, \quad \theta^2 = \mathrm{d}v - uv\mathrm{d}x - s\mathrm{d}y,\\
\omega^1 &= v\mathrm{d}x, \quad \omega^2 = u\mathrm{d}y,\\
\pi^1 &= (1/v)\mathrm{d}p - \frac{u^2 + p}{uv}(\mathrm{d}u - p\mathrm{d}x),\\
\pi^2 &= (1/u)\mathrm{d}s - \frac{v^2 + s}{uv}(\mathrm{d}v - s\mathrm{d}y),
\end{aligned}$$

in the coordinates $x, y, u, v, p = u_x, s = v_y$. The invariants of the characteristic system are $\{x, -u + p/u\}_{\mathrm{func}}$ and $\{y, -v + s/v\}_{\mathrm{func}}$.

Since $u_y = v_x$ we can identify u_y and v_x with the first order derivatives of a function z. The system lifts to a second order equation $z_{xy} = z_x z_y$. This equation has 3 invariants for each of the characteristic subsystems and is therefore contact equivalent to the wave equation.

One can transform the equation $s = pq$ into the wave equation $s = 0$. The symmetry ∂_z used to project to the first order system is then transformed to the scaling symmetry $v = z\partial_z + p\partial_p + q\partial_q + r\partial_r + t\partial_t$. The wave equation can also be projected using the symmetry $w = \partial_z$. The system projects to a first order system given by $u_y = v_x = 0$. We see that are two different projections possible of the wave equation, yielding two non-equivalent first order systems. $\oslash$

Example 11.2.8 (Integration using projection and the method of Laplace). Consider the second order equation (in classical coordinates)

$$s = 2\frac{\sqrt{pq}}{x+y}.$$

The equation is a hyperbolic Goursat equation and is Darboux integrable. The invariants for the two characteristic systems are

$$\left\{x, \frac{r}{\sqrt{p}} + \frac{2\sqrt{p}}{x+y}\right\}_{\text{func}}, \quad \left\{y, \frac{t}{\sqrt{q}} + \frac{2\sqrt{q}}{x+y}\right\}_{\text{func}}.$$

We could integrate this equation using the method of Darboux, but we use a different method here. The method was applied by Goursat [40, Chapitre IX, no. 191] and is an example of a combination of two methods: a symmetry reduction and the method of Laplace.

The equation has ∂_z as a symmetry. The quotient manifold has the structure of a first order system of partial differential equations. If we introduce the coordinates $u = \sqrt{p}$, $v = \sqrt{q}$, then this system is given by

$$u_x = \frac{v}{x+y}, \quad v_y = \frac{u}{x+y}. \tag{11.4}$$

We can eliminate either u or v from these equations. The result is the second order equation

$$u_{xy} + \frac{1}{x+y}u_y - \frac{u}{(x+y)^2} \quad \text{or} \quad v_{xy} + \frac{1}{x+y}v_x - \frac{v}{(x+y)^2},$$

respectively. The (generalized) Laplace invariants of the equation for u are $h = 1/(x+y)^2$ and $k = 0$. Since $k = 0$ we can integrate the equation equation using the method of Laplace. For this particular equation we can write

$$0 = u_{xy} + \frac{1}{x+y}u_y - \frac{u}{(x+y)^2} = \frac{\partial}{\partial y}\left(u_x + \frac{u}{x+y}\right).$$

Hence $u_x + u/(x+y) = B(y)$ and $u = A' + (B - A)/(x+y)$ for two arbitrary functions $A(x)$ and $B(y)$. From the system (11.4) we find $v = B' + (A - B)/(x+y)$. We can easily integrate the solution for u and v to the general solution for z. We find

$$z(x,y) = z(0,0) + \int_{(0,0)}^{(x,y)} \left(A' + \frac{B-A}{x+y} \right)^2 \mathrm{d}x + \left(B' + \frac{A-B}{x+y} \right)^2 \mathrm{d}y.$$

Note that for the general solution z we have

$$\frac{r}{\sqrt{p}} + \frac{2\sqrt{p}}{x+y} = 2u_x + \frac{2u}{x+y} = 2A''(x),$$
$$\frac{t}{\sqrt{q}} + \frac{2\sqrt{q}}{x+y} = 2v_y + \frac{2v}{x+y} = 2B''(y).$$

This gives a functional relation between the invariants in each characteristic system in terms of A'' and B''. Our method yields the same solutions as the method of Darboux.

This example is also discussed in Jurắš [44, Example 4 on p. 20]. He also arrives at a general solution for z using the method of Darboux. His solution is a true general solution in the sense that there are no additional integrations to be carried out. His parameterization has the disadvantage that the independent variables x, y are not used as the parameterization coordinates. $\oslash$

11.3 Future research

This dissertation leaves open a great number of questions. Some are very specific and some have a more general scope. Below we will discuss two points which the author thinks are worth considering for future research.

First there is the concept of pseudosymmetries. In this dissertation we have showed that many familiar concepts (the method of Darboux, pseudoholomorphic curves, Bäcklund transformations, integral extentions) can be understood in the context of pseudosymmetries. Examples of equations that can be analyzed using pseudosymmetries that do not fall into existing methods have also been given. Even though these examples have been given, so far there is mainly a theoretical basis for pseudosymmetries. It is a challenging task to find more examples of pseudosymmetries in applications or even to find and analyze solutions of systems that could not be analyzed using other techniques.

A second problem is the use of computer algebra systems in computations and the development of theory. For this dissertation the use of MAPLE with the packages JETS, VESSIOT and DIFFORMS has been absolutely essential. But even while these packages proved essential, there is a lot of room for improvement. Two problems have could be solved with some time and more efficient computer programs are the following. First there is the class of first order systems for which the image of Nijenhuis tensor has rank 4, but and the two distributions $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal. A brute force calculation of this type of equations might lead to an example of such a system or a proof of the non-existence. A second problem is the calculation of pseudosymmetries of second order scalar equations (Section 9.3). There are obstructions to the existence of such symmetries. For two specific examples these obstructions have been calculated, but in the general case there are obstructions as well.

Appendix A

Various topics

This chapter contains several theorems and topics that have not been included in the main text.

A.1 Pfaffian of a matrix

Let A be an anti-symmetric $2n \times 2n$-matrix. With A we can associate a bi-vector $\omega = a_{ij} e_i \wedge e_j$. We define the *Pfaffian* of A by the equation

$$\frac{1}{n!}\omega^n = \mathrm{Pf}(A) e_1 \wedge \ldots \wedge e_{2n}.$$

For an anti-symmetric $2n \times 2n$-matrix the Pfaffian is equal to the square root of the determinant of the matrix.

Example A.1.1. Let

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Then $\mathrm{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{23}a_{14}$. $\oslash$

Let B be an anti-symmetric bilinear map $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^2$. For a linear map $\xi : \mathbb{R}^2 \to \mathbb{R}$ the composition $\xi \circ B$ is an anti-symmetric bilinear form on $\mathbb{R}^4$. The matrix representation of this bilinear form is an anti-symmetric 4×4-matrix with entries linear in ξ. The Pfaffian of this matrix is a quadratic form in ξ. We say the anti-symmetric bilinear form B is non-degenerate if the discriminant of this quadratic form is non-zero. Every generic anti-symmetric bilinear form on $\mathbb{R}^4$ with values in $\mathbb{R}^2$ is non-degenerate in this sense.

A.2 The relative Poincaré lemma

Let M be a smooth manifold and Φ a homotopy of M with two maps $\phi_0, \phi_1 : M \to M$. To be more precise, Φ is a smooth map

$$\Phi : [0, 1] \times M \to M : (t, x) \mapsto \Phi(t, x)$$

such that

$$\phi_0(t) = \Phi(t, x), \quad \phi_1(t) = \Phi(1, x).$$

Let $\iota : M \to \mathbb{R} \times M$ be defined as $\iota_t(x) = (t, x)$ and let i be the interior product map, i.e., $\mathrm{i}_X(\omega) = X \lrcorner\, \omega$. We define the operator $H : \Omega^k(M) \to \Omega^{k-1}$ by

$$H\alpha = \int_0^1 ({\iota_t}^* \circ \mathrm{i}_{\frac{\partial}{\partial t}} \circ \Phi^*)\alpha \, \mathrm{d}t. \tag{A.1}$$

Theorem A.2.1 (Homotopy lemma). *Let ω be a k-form, Φ a smooth homotopy and the operator H defined as in (A.1). Then*

$${\phi_1}^*\omega = {\phi_0}^*\omega + \mathrm{d}(H\omega) + H(\mathrm{d}\omega). \tag{A.2}$$

Proof. The proof is taken from Duistermaat and Kolk [29, Homotopy Lemma 8.9.5]. Let $\tau_h : \mathbb{R} \times M \to \mathbb{R} \times M : (t, x) \mapsto (t + h, x)$. We have $\phi_t = \Phi \circ \iota_t$ and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}{\phi_t}^*\omega = \frac{\mathrm{d}}{\mathrm{d}t}(\Phi \circ \iota_t)^*\omega &= \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} \iota_{t+h}^*\Phi^*\omega \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} (\tau_h \circ \iota_t)^*\Phi^*\omega = {\iota_t}^*\mathcal{L}_{\frac{\partial}{\partial t}}(\Phi^*\omega) \\
&= {\iota_t}^* \circ [\mathrm{d} \circ \mathrm{i}_{\frac{\partial}{\partial t}} + \mathrm{i}_{\frac{\partial}{\partial t}} \circ \mathrm{d}]\Phi^*\omega \\
&= (\mathrm{d} \circ {\iota_t}^* \circ \mathrm{i}_{\frac{\partial}{\partial t}} \circ \Phi^*)(\omega) + ({\iota_t}^* \circ \mathrm{i}_{\frac{\partial}{\partial t}} \circ \Phi^*)(\mathrm{d}\omega).
\end{aligned}$$

Then

$${\phi_1}^*\omega - {\phi_0}^*\omega = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}{\phi_t}^*(\omega)\,\mathrm{d}t = \mathrm{d}(H\omega) + H(\mathrm{d}\omega).$$ □

Let S be a smooth submanifold of M and write ι_S for the embedding of S in M. A homotopy of M with S is a homotopy Φ such that ϕ_1 is the identity on M and ϕ_0 is a projection of M onto S. In particular there exists a map $\phi : M \to S$ such that $\phi_0 = \psi \circ \iota_S$.

Corollary A.2.2 (Poincaré lemma). *Let ω be a closed form. Then locally ω is exact.*

Proof. This follows from the homotopy formula with Φ a homotopy of a neighborhood with a point. Then $\omega = 0 + \mathrm{d}(H\omega) + H(0) = \mathrm{d}(H\omega)$. □

Lemma A.2.3 (Relative Poincaré lemma). *Let ω be a k-form and I an integrable Pfaffian system for which*

$$\mathrm{d}\omega \equiv 0 \mod I.$$

Then locally there exists a $(k-1)$-form μ such that $\omega \equiv \mathrm{d}\mu \mod I$.

Proof. Let I be generated by the linearly independent 1-forms $\mathrm{d}x^i$, $i = 1, \ldots, n$ on a manifold M. Since the Pfaffian system I is integrable we can consider the leaves of the distribution dual to the $\mathrm{d}x^i$. The quotient B of the manifold by these leaves locally defines a smooth manifold. Let π be the projection $M \to B$ and s a section of the bundle. We can always find a smooth homotopy Φ of M with the image S of the section s such that the leaves of I contract to their intersection with S. Then the homotopy formula implies

$$\omega = {\phi_1}^*\omega = {\phi_0}^*\omega + \mathrm{d}(H\omega) + H(\mathrm{d}\omega).$$

By construction $\phi_0 = s \circ \pi$ and hence ${\phi_0}^*\omega = \pi^*(s^*\omega)$. The form $s^*\omega$ is a 1-form on B and hence $\pi^*(s^*\omega)$ is semi-basic with respect to the bundle $M \to B$. Since the 1-forms in I form a basis for the semi-basic forms of $M \to B$ we see that $\pi^*(s^*\omega) \equiv 0 \mod I$. Let us analyze the term $H(\mathrm{d}\omega)$. Since $\mathrm{d}\omega \equiv 0 \mod I$ we have $\mathrm{d}\omega = \alpha_j \wedge \mathrm{d}x^j$ for certain 1-forms α_j and

$$\begin{aligned} H(\mathrm{d}\omega) &= \int_0^1 ({\iota_t}^* \circ \mathrm{i}_{\frac{\partial}{\partial t}} \circ \Phi^*)(\mathrm{d}\omega)\,\mathrm{d}t \\ &= \int_0^1 {\iota_t}^* \circ \mathrm{i}_{\frac{\partial}{\partial t}} \circ \Phi^*(\alpha_j \wedge \mathrm{d}x^j)\,\mathrm{d}t \\ &= \int_0^1 {\iota_t}^* \circ \mathrm{i}_{\frac{\partial}{\partial t}} (\Phi^*(\alpha_j) \wedge \Phi^*(\mathrm{d}x^j))\,\mathrm{d}t. \end{aligned}$$

Since the x^i are constant along the fibers of $M \to B$, the x^i and $\mathrm{d}x^i$ are invariant under ϕ_t and hence

$$\begin{aligned} H(\mathrm{d}\omega) &= \int_0^1 {\iota_t}^* \circ \mathrm{i}_{\frac{\partial}{\partial t}} (\Phi^*(\alpha_j) \wedge \mathrm{d}x^j)\,\mathrm{d}t \\ &= \int_0^1 {\iota_t}^* \circ \mathrm{i}_{\frac{\partial}{\partial t}} (\Phi^*(\alpha_j))\,\mathrm{d}t \wedge \mathrm{d}x^j \\ &= H(\alpha_j) \wedge \mathrm{d}x^j \equiv 0 \mod I. \end{aligned}$$

Together this implies $\omega \equiv \mathrm{d}(H\omega) \mod I$. □

Corollary A.2.4. *Let ω be a differential 1-form and I an integrable Pfaffian system for which*

$$\omega \not\equiv 0 \mod I, \quad \mathrm{d}\omega \equiv 0 \mod I.$$

Then $\omega \equiv \mathrm{d}z \mod I$ for a smooth non-constant function z.

A.3 Orbits of Lie group actions

In this section we analyze the action of a Lie group G on a smooth manifold. The Lie group is assumed be second-countable, in particular it has at most countably many connected components. We will show that maps into orbits of the Lie group can be lifted to the Lie group itself.

Let G be a Lie group with a smooth action on a manifold M. Let m be a point in M, Gm the orbit through x and G_m the stabilizer of the point m. In formula:

$$Gm = \{ g \cdot m \in M \mid g \in G \}, \quad G_m = \{ g \in G \mid g \cdot m = m \}.$$

The stabilizer group G_m is a closed subgroup of G and $\alpha_m : G/G_m \to M : gG_m \mapsto g \cdot m$ is an immersion with image equal to the orbit Gm. Since we have taken the quotient by the stabilizer group the map α_m is injective and hence the orbit Gm is equal to G/G_m as sets. The quotient G/G_m has the unique structure of a differentiable manifold such that the projection $G \to G/G_m$ defines a principal fiber bundle with structure group G_m [28, Corollary 1.11.5].

The quotient topology of the orbit $Gm = G/G_m$ is not always equal to the subspace topology of $Gm \subset M$. For example, consider the action of $\mathbb{R}$ on the torus $T = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ given by $g \cdot (x, y) = (x + g, y + \sqrt{2}g)$. The orbits are immersed submanifolds. The stabilizer subgroup of a point consists of the identity element. Hence $G_m = \{0\}$ and $Gm \cong G/G_m = G$. The subspace topology is different from the quotient topology. Every open interval in $\mathbb{R}$ defines an open subset in G/G_m, but the image of the interval is not open in the subspace topology on Gm.

For an element $X \in \mathfrak{g}$ we write X_M for the vector field on M defined by the infinitesimal action of G on M. So $X_M(m) = \mathrm{d}/\mathrm{d}t|_{t=0} \exp(tX) \cdot m$. Let $\mathfrak{g}_m = \{ X \in \mathfrak{g} \mid X_M(m) = 0 \}$. Hence $\mathfrak{g}_m$ is equal to the Lie algebra of G_m. Let $\mathfrak{g}(m)$ equal $\{ X_M(m) \mid X \in \mathfrak{g} \}$. The linear space $\mathfrak{g}(m) \subset T_m M$ is equal to the image of G/G_m under $T_{eG_m}\alpha_m$.

Lemma A.3.1. *If $\phi : S \to M$ is a smooth map with image contained in the orbit Gm, then $\phi : S \to G/G_m$ is smooth as well.*

Proof. Suppose that $\phi(s) = m$. Since the orbit is an immersed manifold, there is an open neighborhood U of the identity in G such that U maps onto an open neighborhood of m in the orbit Gm. It is sufficient to show that an open neighborhood of s is mapped by ϕ into the image of U under $g \mapsto g \cdot x$.

Choose a linear complement $\mathfrak{h}$ of $\mathfrak{g}_m$ in $\mathfrak{g}$. In M we choose a submanifold S such that $m \in S$ and $T_m S \oplus \mathfrak{g}_m = T_m M$. From the inverse mapping theorem it follows that there are open neighborhoods $\mathfrak{h}_0$ of 0 in $\mathfrak{h}$ and S_0 of m in S such that the map

$$\mu : \mathfrak{h}_0 \times S_0 \to M : (X, s) \mapsto \exp(X) \cdot s \tag{A.3}$$

is a local diffeomorphism onto an open subset of M. Since $\mu(X, s) \in Gm$ if and only if $s \in Gm$, we have $\mu^{-1}(Gm) = \mathfrak{h}_0 \times (S_0 \cap Gm)$.

Let S_{00} be an open neighborhood of m in S_0 such that the closure of S_{00} as a subset in M is contained in S_0. We want to prove that $(S_{00} \cap Gm)$ is a countable subset of S_{00}.

Let $g_j G_m$ be a sequence in G/G_m that converges to gG_m such that $g_j \cdot m \in S_{00}$. We can choose the elements g_j such that $g_j \to g \in G$. We can restrict $\mathfrak{h}_0$ to an open neighborhood of 0 and choose a neighborhood B of the identity in G_m such that the map $\mathfrak{h}_0 \times B \to G : (X, b) \mapsto \exp(X)b$ is a local diffeomorphism onto a neighborhood of the identity element in G. But then $g^{-1}g_j = \exp(X_j)b_j$ for sequences $X_j \in \mathfrak{h}$ and $b_j \in G_m$ such that $X_j \to 0$ in $\mathfrak{h}_0$ and $b_j \to e$ in G_m as $j \to \infty$. Then $g_j \cdot m = g \cdot \exp(X_j) \cdot b_j \cdot m = g \cdot \exp(X_j) \cdot m = \exp(\mathrm{Ad}\, g(X_j)) \cdot g \cdot m$. The definition of S_{00} implies that the limit $g \cdot m$ of the sequence $g_j \cdot m$ is contained in the closure of S_{00} in M, and hence in S_0. The lineair space $\mathfrak{g}(g \cdot m)$ is equal to $\{ X_M(g \cdot m) \mid X \in \mathfrak{h} \}$. But this space is complementary to $T_{g \cdot m} S_0$. The sequence $g_j \cdot m = \exp(\mathrm{Ad}\, g(X_j)) \cdot g \cdot m$ in S_0 is close to $g \cdot m \in S_0$. This sequence is also in the direction of $\mathfrak{g}(g \cdot m)$ from $g \cdot m$. We conclude that for large j we must have $g_j \cdot m = g \cdot m$.

Hence the collection of $gG_m \in G/G_m$ for which $g \cdot m \in S_{00}$ is a discrete subset of G/G_m. Since G has countably many connected components, G is the union of countably many compact sets. The same is true for G/G_m and we conclude that the collection of $gG_m \in G/G_m$ for which $g \cdot m \in S_{00}$ is countable.

Let $\phi : U \to M$ be a smooth map with image $\phi(U)$ contained in the orbit Gm. Let $u \in U$ and $m = \phi(u)$. We can select an open subset U_0 containing u such that the image of this open subset is contained in the open subset $\mu(\mathfrak{h}_0 \times S_{00})$ constructed above. Let π be the projection of $\mathfrak{h}_0 \times S_{00}$ onto S_{00}. Then $\pi \circ \mu^{-1} \circ \phi$ is a continuous map from U_0 to S_{00} and by assumption the image of this map is contained in $S_{00} \cap Gm$. Since $S_{00} \cap Gm$ is a countable subset of S_{00} we conclude that the image of $\pi \circ \mu^{-1} \circ \phi$ is a single point. Since $m = \phi(u)$ we conclude that $(\pi \circ \mu^{-1} \circ \phi)(U_0) = \{ m \}$ and $(\mu^{-1} \circ \phi)(U_0) \subset \mathfrak{h}_0 \times \{ m \}$. This shows that ϕ defines a smooth map $\phi : U \to G/G_m$. □

A.4 Miscellaneous

The Lie derivative of a differential form can be defined using the exterior derivative and the interior product. Sometimes another definition of the Lie derivative is used and then the definition below is a proposition.

Definition A.4.1 ((H.) Cartan's magic formula, homotopy identity).

$$\mathcal{L}_X \omega = X \lrcorner \mathrm{d}\omega + \mathrm{d}(\omega(X)). \qquad \ominus$$

The relation between the exterior derivative of differential forms and the Lie brackets of vector fields is given by the following proposition.

Proposition A.4.2.

$$\mathrm{d}\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) \tag{A.4}$$

Theorem A.4.3 (Morse lemma). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with a nondegenerate critical point a. Then there exist local coordinates $x^1, \ldots, x^n$ near a such that*

$$f(x) = f(a) + (x^1)^2 + (x^2)^2 + \ldots + (x^k)^2 - (x^{k+1})^2 - \ldots - (x^n)^2.$$

Here k is the index of f at a.

A.5 Grassmannians and conformal actions

In this section we study the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$ and the action by conformal transformations of $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$ on the Grassmannian.

A.5.1 The conformal group

Let $n = p + q$ and consider $\mathbb{R}^n$ with a conformal quadratic form of signature (p, q). With respect to a suitable basis this quadratic form is given by

$$x \mapsto x^T \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix} x.$$

The conformal group $\mathrm{CO}(p, q)$ is the group of linear transformations that preserves this quadratic form up to a scalar factor. We have $\mathrm{CO}(p, q) \cong \mathrm{SO}(p, q) \times H$, where $H = \mathbb{R}^*$ is the group of scalar multiplications (which are sometimes called *homotheties* in this context). The dimension of $\mathrm{CO}(p, q)$ is $n(n-1)/2 + 1$. In the special case $p = q = 2$ we have $\mathrm{CO}(2, 2) \cong \mathrm{SL}(2) \times \mathrm{SL}(2) \times H$. See Akivis and Goldberg [1]

A.5.2 Local coordinates

Let V be a vector space of dimension 4 and and consider the action of the group $G = \mathrm{GL}(V)$ on V. The action of G on V induces an action on the Grassmannian $M = \mathrm{Gr}_2(V)$ and this action is transitive. We will analyze the orbits of this action on the tangent space TM.

We identify V with $\mathbb{R}^4$. We introduce local coordinates for the Grassmannian in the form of 2×2-matrices A. The matrix A corresponds to the 2-plane spanned by vectors in the image of

$$\begin{pmatrix} I \\ A \end{pmatrix}.$$

Alternatively, we can define the 2-plane corresponding to A as the inverse image of $\mathbb{R}^2 \times \{0\}$ under the map $\tilde{A} \in \mathrm{GL}(4, \mathbb{R})$ defined by

$$\tilde{A} = \begin{pmatrix} I & 0 \\ -A & I \end{pmatrix}.$$

The group G acts on first representation by multiplication on the left. On $\tilde{A}$ an element $g \in G$ acts by multiplication on the right with the g^{-1}. The action of $g \in \mathrm{GL}(V)$ on the Grassmannian in local coordinates is given by

$$g = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} : A \mapsto (\tilde{c} + \tilde{d}A)(\tilde{a} + \tilde{b}A)^{-1}.$$

We let x_0 be the point in the Grassmannian that corresponds to $A = 0$. The stabilizer group H of x_0 is equal to the set of matrices

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{d} \end{pmatrix}, \tag{A.5}$$

with $\tilde{a}$, $\tilde{d}$ invertible 2×2-matrices and $\tilde{b}$ an arbitrary 2×2-matrix.

A.5.3 Action on the tangent space

We want to know how the stabilizer H acts on the tangent space of the Grassmannian. Suppose that

$$t \mapsto \begin{pmatrix} I \\ tX \end{pmatrix}$$

is a curve through the point x_0 that represents a tangent vector in the Grassmannian. The group H acts on this curve as

$$t \mapsto \begin{pmatrix} \tilde{a} + \tilde{b}tX \\ \tilde{d}tX \end{pmatrix}.$$

The matrix represents an element in the Grassmannian and an equivalent representation is

$$t \mapsto \begin{pmatrix} I \\ \tilde{d}tX(\tilde{a} + \tilde{b}tX)^{-1} \end{pmatrix} \cong \begin{pmatrix} I \\ \tilde{d}tX\tilde{a}^{-1} + \mathcal{O}(t^2) \end{pmatrix}.$$

So the action of H on the tangent space in local coordinates is given by

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{d} \end{pmatrix} \cdot X = \tilde{d}X\tilde{a}^{-1}. \tag{A.6}$$

A.5.4 Conformal isometry group of the Grassmannian

In this section we will show that the conformal isometry group of the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$ is given by $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$. The discussion in this section gives the proof of Lemma 2.1.5. For the conformal isometries of the oriented Grassmannian the same reasoning holds, with minor changes to the groups involved.

Let M be a manifold of dimension n with a conformal structure of signature (p, q). For $n > 3$ a conformal transformation is completely determined by the action on the second order jets. In particular if ϕ_1, ϕ_2 are two conformal transformations with $j^2\phi_1(x) = j^2\phi_2(x)$ for a point x in M, then $\phi_1 = \phi_2$ on a neighborhood of x. See Kobayashi [46].

We will show that the group $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$ acts transitively on the second order jets of conformal transformations. This proves that for every conformal transformation ϕ of the Grassmannian there is a transformation ψ induced from $A \in \mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$ such that $\psi^{-1} \circ \phi$ is equal to the identity on the 2-jets and hence equal to the identity on an open neighborhood. Since $\mathrm{Gr}_2(\mathbb{R}^4)$ is connected it follows that $\psi = \phi$.

The action of the group $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$ on the Grassmannian is transitive. Hence we can consider the subgroup H of transformations in $\mathbb{P}\,\mathrm{GL}(4, \mathbb{R})$ that fix a point in the Grassmannian. In the previous section we have seen that these transformations are represented by the matrices of the form (A.5). The action of H on the tangent space of the Grassmannian is given by $X \mapsto \tilde{d}X\tilde{a}^{-1}$. This is an injective representation of $\tilde{H} = \mathrm{GL}(2, \mathbb{R}) \times$

$\mathrm{GL}(2,\mathbb{R})/\mathbb{R}^* \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^*$. The 1-jets of conformal transformations fixing a point are a subset of the conformal transformations $\mathrm{CO}(2,2)$. This group is isomorphic to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^*$, see Section A.5.1. Since $\tilde{H}$ has the same dimension as $\mathrm{CO}(2,2)$, the same number of connected components and $\tilde{H} \subset \mathrm{CO}(2,2)$ we conclude that $\tilde{H}$ acts transitively on the 1-jets of conformal transformations.

Finally we have to consider the action on the second order jets. The full group $\mathbb{P}\,\mathrm{GL}(4,\mathbb{R})$ has dimension 15. The dimension of the subgroup $\mathbb{P}\,\mathrm{GL}(4,\mathbb{R})$ fixing a point and a 1-jet is 4. This is equal to the dimension of the Lie algebra prolongation of $\mathfrak{co}(2,2)$ (see McKay [53, p. 29]). Since the Lie algebra prolongation is connected, we can conclude that the group $\mathbb{P}\,\mathrm{GL}(4,\mathbb{R})$ acts transitively on the 2-jets of conformal transformations.

A.5.5 Orbits

In this section we consider the action of the conformal group $\mathrm{CO}(2,2) \cong (\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R}))/\mathbb{R}^* \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^*$ on the space W of 2×2-matrices given by

$$(\tilde{a}, \tilde{d}) \cdot X \mapsto \tilde{d} X \tilde{a}^{-1}.$$

This action is a conformal action in the sense that the action preserves the conformal quadratic form $X \mapsto \det(X)$. The conformal group is described in Bryant et al. [13, Section 7.1, Case 2, p. 272] and in McKay [51, p. 9].

The conformal action induces an action on the linear subspaces of dimension 2 in W. This action has 5 orbits, representatives for the 5 orbits are given by

$$\begin{aligned}
W_1 &= \{ \begin{pmatrix} 0 & 0 \\ x^1 & x^2 \end{pmatrix} \mid x^i \in \mathbb{R} \}, \\
W_2 &= \{ \begin{pmatrix} 0 & x^1 \\ 0 & x^2 \end{pmatrix} \mid x^i \in \mathbb{R} \}, \\
W_3 &= \{ \begin{pmatrix} x^1 & x^2 \\ x^2 & 0 \end{pmatrix} \mid x^i \in \mathbb{R} \}, \\
W_4 &= \{ \begin{pmatrix} x^1 & 0 \\ 0 & x^2 \end{pmatrix} \mid x^i \in \mathbb{R} \}, \\
W_5 &= \{ \begin{pmatrix} x^1 & -x^2 \\ x^2 & x^1 \end{pmatrix} \mid x^i \in \mathbb{R} \}.
\end{aligned}$$

The last two orbits are open and represent the hyperbolic planes and elliptic planes, respectively. For example on W_4 the conformal quadratic form det takes the form $x_1 x_2$ (signature $(1,1)$) and on W_5 the conformal quadratic form takes the form $(x_1)^2 + (x_2)^2$ (signature $(2,0)$).

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Subject index

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Samenvatting

Dit proefschrift gaat over het gebruik van (pseudo)symmetrieën in het oplossen en begrijpen van partiële differentiaalvergelijkingen.

Symmetrieën en pseudosymmetrieën

In het dagelijks leven komen we veel symmetrieën tegen. Veel voorbeelden van symmetrieën zijn te vinden in het werk van de Nederlandse kunstenaar M.C. Escher (1898 – 1972). In Figuur D.1(a) staat een voorbeeld van een afbeelding met symmetrie. De symmetrieën in deze figuur zijn de translaties. Er zijn diagonale en verticale translaties mogelijk. We kunnen een symmetrie uitdelen. We houden dan een deel van de oorspronkelijke afbeelding over die in zekere zin alle informatie bevat. In Figuur D.1(b) is het quotiënt van de hele afbeelding onder de symmetrie weergegeven. Zo'n quotiënt wordt ook wel een fundamenteel domein genoemd. Omgekeerd, als een fundamenteel domein en het type van de symmetriegroep gegeven zijn, dan kunnen we de oorspronkelijke afbeelding reconstrueren.

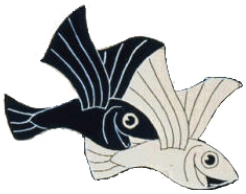
We kunnen nog een stap verder gaan. De zwarte en de witte vissen in de figuur hebben allemaal dezelfde vorm. Als we enkel op de vorm letten en niet op de kleur, dan is translatie over de afstand van één vis ook een symmetrie. We noemen deze vorm van symmetrie een *pseudosymmetrie* omdat maar een deel van de aanwezige structuur (namelijk de vorm) is behouden en een ander deel (namelijk de kleur) verloren gaat.

De pseudosymmetrieën in dit proefschrift gaan nog een stap verder. Bekijk Figuur 9.1 op pagina 202. Hierin is een vectorveld getekend (de zwarte pijlen) die een symmetrie hebben in de horizontale richting. In de verticale richting is er geen symmetrie: de pijlen op dezelfde verticale lijn staan in een verschillende richting. Kijken we echter enkel naar de component van het vectorveld loodrecht op de translatierichting, dan is wel sprake van een symmetrie. De horizontale component van de pijlen op een verticale lijn is altijd even groot. Dus translatie in de verticale richting geeft een pseudosymmetrie van het vectorveld. Belangrijk hierbij is dat de structuur die behouden wordt, namelijk de horizontale component, afhangt van de keuze van de pseudosymmetrie (verticale translaties).

Partiële differentiaalvergelijkingen

Partiële differentiaalvergelijkingen worden gebruikt in allerlei takken van de wetenschap om modellen op te stellen. Voorbeelden zijn de bewegingsvergelijkingen uit de klassieke mecha-

(a) Figuur met symmetrie



(b) Quotiënt

(c) Quotiënt onder pseudosymmetrie

Figuur D.1: Symmetrieën. All M.C. Escher Works © 2006 The M.C. Escher Company – the Netherlands. All rights reserved. Used by permission. http://www.mcescher.com

nica, reactie-diffusie vergelijkingen uit de chemie en populatie modellen uit de mathematische biologie.

Als voorbeeld kan men de beweging van een blad aan de oppervlakte van een stromende rivier bekijken. De positie van het blad wordt geven door middel van twee functies $x_1(t), x_2(t)$ die van de tijd t afhangen. De stroming van de rivier wordt beschreven door op elke plaats de snelheid en de richting van de stroming te geven. De stroming wordt dus gegeven door een vectorveld $v(x) = (v_1(x), v_2(x))$ dat van de plaats afhangt. De afgeleide $\mathrm{d}x/\mathrm{d}t$ beschrijft de verandering van de plaats van het blad. Deze is gelijk aan de snelheid v waarmee de rivier stroomt op de plaats waar het blad zich op dat moment bevindt.

In de wiskunde worden dit soort relaties beschreven met behulp van *differentiaalvergelijkingen*. De differentiaalvergelijking die de beweging van het blad beschrijft, wordt gegeven door

$$x' = v(x).$$

Een oplossing van deze differentiaalvergelijking is een functie $x(t)$ die voldoet aan de relatie $x'(t) = v(x(t))$.

In ons voorbeeld kon de differentiaalvergelijking beschreven worden met behulp van een vectorveld. Het vectorveld schrijft voor in welke richting de oplossingen mogen bewegen. Meer algemene partiële differentiaalvergelijkingen kunnen we beschrijven door middel van *contact distributies*. Een distributie wordt gegeven door een verzameling van vectorvelden. Voor elk punt geven deze vectorvelden de verschillende richtingen aan waarin een oplossing van de differentiaalvergelijking mag bewegen.

Veel partiële differentiaalvergelijkingen die van belang zijn in allerlei takken van de wetenschap bezitten symmetrieën. De Noorse wiskundige Sophus Lie (1842 – 1899) was één van de eerste wiskundigen die symmetrieën van differentiaalvergelijkingen gebruikte om de vergelijkingen op te lossen. Ook partiële differentiaalvergelijkingen kunnen we uitdelen naar een symmetrie. Het quotiënt is een nieuwe partiële differentiaalvergelijkingen die vaak eenvoudiger is dan de oorspronkelijke vergelijking. Door deze nieuwe vergelijkingen op te lossen, kan men vaak ook oplossingen vinden van de oorspronkelijke vergelijking.

Ook pseudosymmetrieën kunnen gebruikt worden om partiële differentiaalvergelijkingen te reduceren tot nieuwe vergelijkingen. Dit proces is echter minder eenvoudig dan voor gewone symmetrieën. Omdat pseudosymmetrieën geen exacte symmetrieën zijn (er gaat informatie verloren), moet uitgezocht worden welke informatie men nodig heeft om ervoor te zorgen dat bij het uitdelen een nieuwe partiële differentiaalvergelijking ontstaat en deze ook nuttig is om de oorspronkelijke vergelijking mee te bestuderen.

Inhoud

We geven nu een kort overzicht van de wiskundige inhoud van het proefschrift. Het proefschrift gaat over de *contact structuren* van partiële differentiaalvergelijkingen. Deze contact structuren geven een meetkundige beschrijving van de vergelijkingen in termen van het opspansel van een aantal vectorvelden. Het opspansel van een aantal vectorvelden wordt ook wel een *distributie* genoemd. De belangrijkste klassen van partiële differentiaalvergelijkingen in dit proefschrift zijn de eerste orde stelsels van twee vergelijkingen voor twee functies van twee variabelen en de tweede orde scalaire vergelijkingen in twee variabelen. Door de lage dimensies van deze twee klassen ontstaat een hele rijke meetkundige structuur.

Het proefschrift begint met Hoofdstuk 1 waarin de notatie wordt vastgelegd en een aantal definities en begrippen worden ingevoerd die in de rest van het proefschrift nodig zijn.

In Hoofdstuk 2 worden vervolgens hyperbolische getallen en hyperbolische oppervlakken besproken. De hyperbolische getallen bestaan uit de elementen $(x, y) \in \mathbb{R}^2$ met vermenigvuldiging $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, y_1y_2)$. Deze hyperbolische getallen vervullen de rol voor hyperbolische vergelijkingen die de complexe getallen hebben voor elliptische vergelijkingen. Ook worden oppervlakken in de Grassmanniaan van 2-dimensionale lineaire deelruimten van een 4-dimensionale vectorruimte bestudeerd. Deze oppervlakken zijn van belang voor de microlokale analyse van eerste orde stelsels.

In de hoofdstukken 3, 4, 5 en 6 wordt een structuurtheorie ontwikkeld voor eerste orde stelsels en tweede orde vergelijkingen. De stelling van Vessiot (pagina 82) geeft een exacte beschrijving van de contact structuur van tweede orde scalaire partiële differentiaalvergelijkingen. Voor eerste orde stelsels is er een vergelijkbare stelling (Theorem 4.6.4). De structuurtheorie wordt vervolgens gebruikt om invarianten van differentiaalvergelijkingen mee te bepalen en verbanden te leggen met het werk van Gardner and Kamran [38], Juráš [44] en McKay [51].

Één van de structuren die wordt gevonden in Hoofdstuk 4 voor elliptische eerste orde stelsels is een bijna-complexe structuur op de contact distributie. Het blijkt dat we de vergelijkingen voor pseudoholomorfe krommen voor een bijna-complexe structuur exact kunnen beschrijven met behulp van de Nijenhuis tensor voor eerste orde stelsels. Dit wordt beschreven in Hoofdstuk 7. Voor tweede orde vergelijkingen leidt een zelfde analyse van de Nijenhuis tensor tot de Monge-Ampère vergelijkingen.

In Hoofdstuk 9 wordt een definitie gegeven van pseudosymmetrieën voor distributies, voor eerste orde stelsels en voor tweede orde vergelijkingen. Er wordt een methode beschreven (ontwikkeld samen met Robert Bryant) om deze pseudosymmetrieën efficiënt uit te rekenen voor tweede orde vergelijkingen. Ook wordt het verband gelegd tussen pseudosymmetrieën en de begrippen integreerbare uitbreiding en Bäcklund transformatie.

In de hoofdstukken 8 en 10 bestuderen we Darboux integreerbaarheid. De methode van Darboux is in de 19e eeuw geïntroduceerd door Gaston Darboux. Het oplossen van een eerste orde stelsel of tweede orde vergelijking die voldoende invarianten bevat voor beide karakteristieke systemen, wordt hiermee gereduceerd tot het oplossen van gewone differentiaalvergelijkingen. De methode van Darboux is een speciaal geval van een pseudosymmetrie waarbij het quotiënt een bijzonder eenvoudige structuur heeft. Generalisaties van de methode van Darboux waarbij het quotiënt een iets minder mooie structuur heeft, worden beschreven onder de naam vector pseudosymmetrieën in Hoofdstuk 11. Het quotiënt is dan een systeem voor pseudoholomorfe krommen voor een bijna-complexe of bijna-product structuur.

Tot slot wordt in Hoofdstuk 11 een overzicht gegeven van de verschillende projectie methoden die behandeld zijn. Ook wordt aangegeven wat mogelijkheden zijn voor toekomstig onderzoek.

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Curriculum vitae

Pieter Thijs Eendebak was born on September 10th 1979 in Wageningen, The Netherlands. He grew up in Oosterbeek and attended elementary education there. From 1991 to 1997 he studied at the Katholiek Gelders Lyceum where he obtained his VWO.

In 1997 he started his study in both mathematics and physics at Utrecht University. This resulted in his propaedeuse Physics in 1998, his propaedeuse Mathematics in 1998 and his propaedeuse Computer Science in 1999. During his studies he became more and more interested in mathematics. In 2002 he obtained his Master degree in Mathematics (cum laude) under supervision of prof.dr. J.J. Duistermaat. In 2003 he obtained his Master degree in Physics (cum laude) as well.

In 2002 he started his Ph.D. research on the geometric treatment of partial differential equations. During his Ph.D.-studies he attended seminars by the MRI (Mathematical Research Institute) and visited the University of South Florida, where he taught a course on introductory statistics. Also he taught a variety of undergraduate mathematics tutorials at Utrecht University. His research resulted in this dissertation.