Contact Structures and Projection Methods

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November 1, 2006

Outline

- Distributions (vector field systems)
- Vessiot theorem for second order scalar equations
- Structure theory (analysis of integral elements)
- Projection methods
  - Darboux integrability
  - Symmetry reductions

Distributions

Definition

Let M be a smooth manifold. A *distribution* on M is a smooth constant rank vector subbundle of the tangent bundle TM.

Let $\mathcal{V}$ be a distribution. For a vectorfield X we write $X \subset \mathcal{V}$ if for all points $m \in M$ we have $X_m \in \mathcal{V}_m$.

For example on $\mathbb{R}^3$ with coordinates x, y, z we can define the rank two distribution $\mathcal{V}$ spanned by the vector field ∂_x and ∂_y. Then

$$\mathcal{V}_{x,y,z} = \{ X = a\partial_x + b\partial_y \mid a, b \in \mathbb{R} \}.$$

Then $\partial_x + z\partial_y \subset \mathcal{V}$.

The Lie brackets

For any pair of vector fields X, Y on M the Lie bracket $[X, Y]$ is again a vector field on M. Definition:

$$[X, Y] = \mathcal{L}_X Y = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)_* Y.$$

The value of $[X, Y]$ at a point M depends not only on the values of X, Y at m, but also on the first order derivatives of X, Y at the point m.

Lemma

Let X, Y be vector fields in the distribution $\mathcal{V}$. Then $[X, Y]_m$ mod $\mathcal{V}_m$ *depends only on the values X_m, Y_m.*
Hence the Lie brackets induce a tensor

$$\lambda : \mathcal{V} \times_M \mathcal{V} \to TM/\mathcal{V}.$$

We call this map the Lie brackets modulo the subbundle.

▸ Skip proof

Proof

Proof.

Let $X, Y \subset \mathcal{V}$ and assume that at the point m we have $X_m = 0$. Since the Lie brackets are bilinear and anti-symmetric it is sufficient to prove that $[X, Y]_m \equiv 0 \mod \mathcal{V}$.

Assume $X_m = 0$. Then we can write $X = \phi^j Z_j$ with $Z_j \subset \mathcal{V}$ and $\phi^j(m) = 0$. Then modulo $\mathcal{V}$

$$[X, Y] \equiv [\phi^j Z_j, Y] \equiv \phi^j [Z_j, Y] - Y(\phi^j) Z_j \equiv \phi^j [Z_j, Y]$$

Hence at m we have $[X, Y]_m \equiv 0 \mod \mathcal{V}$. □

Some definitions

- We define the *derived bundle* $\mathcal{V}'$ of $\mathcal{V}$ as the distribution spanned by $\mathcal{V}$ and the image of the Lie brackets modulo $\mathcal{V}$. Equivalently,

$$\mathcal{V}' = \{ [X, Y]_m \mid X, Y \subset \mathcal{V} \}.$$

- The kernel of the Lie brackets modulo $\mathcal{V}$ is called the *Cauchy characteristic space* $C(\mathcal{V})$. The Cauchy characteristic space has the property that $X \subset C(\mathcal{V})$ if and only if $[X, Y] \subset \mathcal{V}$ for all $Y \subset \mathcal{V}$.
- An *integral manifold* U of a distribution $\mathcal{V} \subset TM$ is a submanifold of M such that the tangent space $T_m U \subset \mathcal{V}_m$ for all points in m.
- An invariant of a distribution $\mathcal{V}$ is a function ϕ such that $X(\phi) = 0$ for all $X \subset \mathcal{V}$.

Frobenius theorem

Theorem (Frobenius theorem)

Suppose $\mathcal{V}$ is a rank k distribution such that $[X, Y] \subset \mathcal{V}$ for all $X, Y \subset \mathcal{V}$. Then locally there is a foliation of M by integral manifolds of dimension k.

The maximal integral manifolds are called the *leaves* of $\mathcal{V}$.

Solutions of partial differential equations and integral manifolds

We will consider second order scalar partial differential equations in two independent variables. The independent variables are always x, y, the dependent variable z and the first and second order derivatives will be written as

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y},$$
$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

The *graph* of any function $z(x, y)$ is a 2-dimensional submanifold of the zeroth order jet bundle. The graph of the second order jet of the function is a submanifold of $Q = \mathrm{J}^2(\mathbb{R}^2, \mathbb{R})$.

Example: jet bundle

We consider the function $z(x, y) = x^2 + y$.
Let $X = \mathrm{J}^0(\mathbb{R}^2)$ be the zero-th order jet bundle. The graph of this function is given by

$$\{ (x, y, z(x, y)) = (x, y, x^2 + y) \in \mathrm{J}^0(\mathbb{R}^2) \mid x, y \in \mathbb{R} \} \subset X.$$

The second order jet of $z(x, y)$ is given by the map

$$\mathbb{R}^2 \mapsto \mathrm{J}^2(\mathbb{R}^2) : (x, y) \mapsto (x, y, z, z_x, z_y, \dots, z_{yy}).$$

The graph of the second order jet $j^2 z$ is the submanifold of $\mathrm{J}^2(\mathbb{R}^2)$ given by

$$U = \{ m = (x, y, x^2 + y, 2x, 1, 2, 0, 0) \in \mathrm{J}^2(\mathbb{R}^2) \mid x, y \in \mathbb{R} \}.$$

The tangent space to such a graph is spanned by two vectors

$$
\begin{aligned}
&(x, y) \mapsto (x, y, z, p, q, r, s, t) \\
&(1, 0, \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial x \partial y^2}) \\
&\quad = (1, 0, p, r, s, \frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial x \partial y^2}), \\
&(0, 1, q, s, t, \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial x \partial y^2}, \frac{\partial^3 z}{\partial y^3}).
\end{aligned}
$$

These vectors are contained in the *contact distribution* $\mathcal{W}$ spanned by

$$
\begin{aligned}
X &= \partial_x + p\partial_z + r\partial_p + s\partial_q, \\
Y &= \partial_y + q\partial_z + s\partial_p + t\partial_q, \\
R &= \partial_r, \quad S = \partial_s, \quad T = \partial_t.
\end{aligned}
$$

The distribution $\mathcal{W}$ defines the *contact structure* on the jet bundle.

In other words: the graph of the function $z(x, y)$ defines an integral manifold of $\mathcal{W}$. The converse is also true.

Theorem

Let $(Q, \mathcal{W})$ be the second order jet bundle of X with contact distribution defined previously. Then an integral manifold of $\mathcal{W}$ that is transversal to the projection $Q \rightarrow X$ is locally equal to the graph of the 2-jet a function $z(x, y)$.

Partial differential equations

Any second order equation $F(x, y, z, p, q, r, s, t) = 0$ defines a hypersurface $M \subset Q$. On M we define a contact structure by $\mathcal{V} = \mathcal{W} \cap TM$.

A solution of the partial differential equation $F = 0$ is a function $z(x, y)$ for which the graph of the 2-jet is a submanifold of M. At the same time the graph is an integral manifold of $\mathcal{W}$ and hence of $\mathcal{V}$.

Theorem

Locally there is a one-to-one correspondence between integral manifolds of $(M, \mathcal{V})$ transversal to the projection $M \to X$ and solutions of the partial differential equation $F = 0$.

Vessiot theorem

Theorem

Let M be a 7-dimensional manifold with a rank 4 distribution $\mathcal{V}$. Then $(M, \mathcal{V})$ is locally equivalent to the equation manifold of a second order scalar equation in two independent variables if and only if

- *For every $m \in M$ the Cauchy characteristic space $C(\mathcal{V})_m$ of $\mathcal{V}$ at m is equal to zero.*
- *For every $m \in M$ the derived bundle $\mathcal{V}'_m$ has rank 6.*
- *For every $m \in M$, $C(\mathcal{V}')_m$ is contained in $\mathcal{V}_m$ and has rank 2.*

Ernest Vessiot

Ernest Vessiot (1865–1952)

- Picard-Vessiot theory (differential Galois theory), ballistics
- Formulation of partial differential equations in terms of distributions (dual to the work of Cartan), Darboux integrable hyperbolic Goursat equations

Contact distribution for the wave equation

Consider the wave equation $z_{xy} = s = 0$. We can use x, y, z, p, q, r, t as coordinates for the equation manifold. The contact distribution is spanned by the vectors

$$F_1 = \partial_x + p\partial_z + r\partial_p, \quad F_2 = \partial_r,$$
$$G_1 = \partial_y + q\partial_z + r\partial_q, \quad G_2 = \partial_t.$$

Note that $[F_1, F_2] = -\partial_p$, so ∂_p is in the derived bundle. One can easily check that $\mathcal{V}' = \mathrm{span}(F_1, F_2, G_1, G_2, \partial_p, \partial_q)$.
In a similar way $C(\mathcal{V}) = 0$, $C(\mathcal{V}') = \mathrm{span}(F_2, G_2)$.

Integral elements

- Let U be an integral manifold of $\mathcal{V}$, i.e., $TU \subset \mathcal{V}$. Then for any pair of vector fields X, Y on U we have $[X, Y] \subset TU$. Therefore the Lie brackets modulo $\mathcal{V}$ must vanish on TU.
- The subspaces of $\mathcal{V}_m$ on which the Lie brackets modulo $\mathcal{V}$ vanish are called the *integral elements* of $\mathcal{V}$. These integral elements are the solutions of the linearization of the equation.

The Lie brackets modulo $\mathcal{V}$

Since the Lie brackets are contained in $\mathcal{V}'$ (by definition), the Lie brackets define an anti-symmetric bilinear map

$$\lambda : \mathcal{V} \times_M \mathcal{V} \to \mathcal{V}'/\mathcal{V}.$$

For any form $\xi \in (\mathcal{V}'/\mathcal{V})^*$ we can consider the 2-form $(\xi \circ \lambda) : \Lambda^2\mathcal{V} \to \mathbb{R}$ and the 4-form

$$(\xi \circ \lambda) \wedge (\xi \circ \lambda) : \Lambda^4\mathcal{V} \to \mathbb{R}.$$

We have a conformal quadratic form on $\mathcal{V}'/\mathcal{V}$.

- Discriminant of quadratic form: type of equation
- We choose two isotropic elements $\xi_\pm$ ('roots'). Then

  $$\xi_\pm \circ \lambda$$

  are anti-symmetric bilinear forms on $\mathcal{V}$. The kernels $\mathcal{V}_\pm$ are called the *Monge systems*.

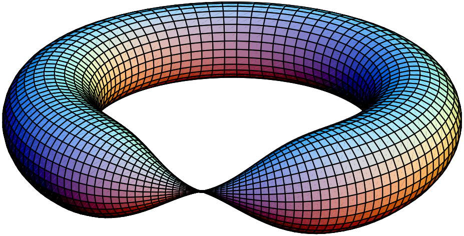
Integral elements

Hyperbolic For hyperbolic equations we have $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$. The splitting of $\mathcal{V}$ defines a *direct product structure*. The 2-dimensional integral elements are precisely those linear subspaces that have non-zero intersection with both $\mathcal{V}_\pm$. The space of integral elements at m is a torus

Elliptic For elliptic equations the isotropic elements are complex and hence we find complex Monge systems $\mathcal{V} \otimes \mathbb{C} = \mathcal{V}_+ \oplus \mathcal{V}_-$. We define $J : \mathcal{V} \otimes \mathbb{C}$ by $J|_{\mathcal{V}_\pm} = \pm i$. Then J restricted to $\mathcal{V}$ defines a *complex structure*. The 2-dimensional integral elements are complex lines for J.

Parabolic Between elliptic and hyperbolic

Integral elements for a parabolic equation



Geometry of partial differential equations

	PDE	Contact structure
framework	local coordinates	geometric
system	system of PDE's	distribution
solutions	functions	integral manifolds
differentiation	partial derivatives	Lie brackets, d operator

Contact distribution for the wave equation

Consider the wave equation $s = 0$. We can use x, y, z, p, q, r, t as coordinates for the equation manifold. The Monge systems are given by

$$\mathcal{V}_+ = \mathrm{span}(\partial_x + p\partial_z + r\partial_p, \partial_r),$$
$$\mathcal{V}_- = \mathrm{span}(\partial_y + q\partial_z + t\partial_q, \partial_t).$$

The invariants of $\mathcal{V}_-$ are x, p, r and the invariants of $\mathcal{V}_+$ are y, q, t.

Gaston Darboux

Gaston Darboux (1842–1917)

- Darboux integral (integration theory), Darboux theorem (symplectic geometry)
- The method of Darboux to integrate second order partial differential equations

The method of Darboux

- Find invariants of the Monge systems
- Make a projection
- Lift pseudoholomorphic curves to solutions of the equation

Step 1: If each Monge system has at least two invariants the equation is Darboux integrable.

Intermezzo: almost complex and almost product structures

Almost complex structure: $J : TM \to TM$, $J^2 = -1$

Almost product structure: $K : TM \to TM$, $K^2 = 1$
The eigenspaces of K for the eigenvalues ± 1 give the tangent space a direct sum structure $T_m M = \mathcal{V}_+ \oplus \mathcal{V}_-$.

The method of Darboux 2: projection

Suppose I^1, I^2 are invariants of $\mathcal{V}_-$ and I^3, I^4 invariants of $\mathcal{V}_+$.
Define the *Darboux projection* $\pi : M \to \mathbb{R}^2 \times \mathbb{R}^2 : m \mapsto I^j(m)$.
Then $T\pi_*(\mathcal{V}_+) \subset T(\mathbb{R}^2 \times \{ 0 \})$ and $T\pi_*(\mathcal{V}_-) \subset T(\{ 0 \} \times \mathbb{R}^2)$.
The projection is transversal to $\mathcal{V}$ and hence $T_m\pi : \mathcal{V}_m \to T_{\pi(m)}B$ is injective.

The projection intertwines the direct product structure on $\mathcal{V}$ with the direct product structure on $\mathbb{R}^2 \times \mathbb{R}^2$, i.e.,

$$T\pi \circ K = K^B \circ T\pi$$

The method of Darboux: elliptic equations

For an elliptic equation we can take two complex invariants of one of the complex Monge systems. Then we can define a projection $M \to \mathbb{C}^2$ such that

$$T\pi \circ J = J^B \circ T\pi$$

Here J^B is the complex structure on $\mathbb{C}^2$.

Intermezzo: pseudoholomorphic curves

For a complex structure J a pseudoholomorphic curve is a real 2-dimensional manifold for which the tangent space is J-invariant.
Construction for $\mathbb{C}^2$: pseudoholomorphic curves are given by the graphs of complex-differentiable functions $\mathbb{C} \to \mathbb{C}$.

For a hyperbolic structure $K : TM \to TM$ a (hyperbolic) pseudoholomorphic curve is a real 2-dimensional submanifold S with K-invariant tangent space. We require the additional condition that the intersection of TS with the eigensystems of K is non-trivial.
Construction for $\mathbb{R}^2 \times \mathbb{R}^2$: choose two curves ϕ_1, ψ_2 in $\mathbb{R}^2$. Then the direct product $\tilde{S} = \phi_1 \times \phi_2 \subset \mathbb{R}^2 \times \mathbb{R}^2$ defines a pseudoholomorphic curve.

The method of Darboux 3: lifting solutions

Let $S = \pi^{-1}(\tilde{S})$ and on S we define $\mathcal{W} = \mathcal{V} \cap TS$. Since $\mathcal{V}$ is transversal to π we have $\operatorname{rank} \mathcal{W} = \dim \tilde{S} = 2$.

For any pair of vector fields $X, Y \subset \mathcal{W}$ we have

- $T\pi([X, Y]) = [T\pi(X), T\pi(Y)] \subset T\tilde{S}$. Hence $[X, Y] \subset TS$.
- Assume $X \subset \mathcal{W}_+ = \mathcal{V}_+ \cap TS$, $Y \subset \mathcal{W}_- = \mathcal{V}_- \cap TS$. Then $[X, Y] \subset [\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V}$.
  Since $\mathcal{W}$ has rank 2 this shows that $[X, Y] \subset \mathcal{V}$ for all $X, Y \subset \mathcal{V}$.

Together: $[X, Y] \subset \mathcal{W} = TS \cap \mathcal{V}$.

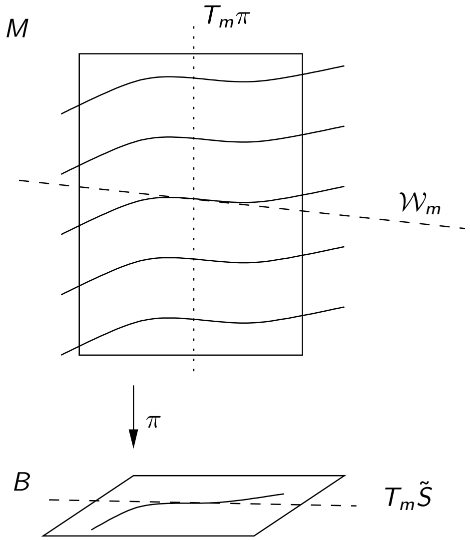
The method of Darboux 3: lifting solutions

Hence the distribution $\mathcal{W}$ is integrable and by the Frobenius theorem there exists a local foliation of M by 2-dimensional integral manifolds of $\mathcal{W}$ and hence of $\mathcal{V}$.

Theorem

There is a one-to-one correspondence between pseudoholomorphic curves for the projected structure J^B and 3-dimensional families of integral manifolds of $\mathcal{V}$.

Geometric picture of lifting



Example: $s = p/(y-x)$

Let us consider the second order scalar partial differential equation $s = p/(y-x)$. The Monge systems are given by

$$\mathcal{V}_+ = \mathrm{span}(\partial_x + p\partial_z + r\partial_p + \frac{p}{y-x}\partial_q, \partial_r),$$

$$\mathcal{V}_- = \mathrm{span}(\partial_y + q\partial_z + \frac{p}{y-x}\partial_p + t\partial_q + \left(\frac{r(y-x)+p}{(y-x)^2}\right)\partial_r, \partial_t).$$

Invariants The distribution $\mathcal{V}_+$ has invariants y and $\tau = t$. The distribution $\mathcal{V}_-$ has invariants x, $p/(y-x)$ and $\rho = r/(y-x) + p/(y-x)^2$;
Projection We make the projection $\pi : M \to \mathbb{R}^2 \times \mathbb{R}^2$. On $\mathbb{R}^2 \times \mathbb{R}^2$ we use (x, ρ) and (y, τ) as coordinates.

Example: $s = p/(y-x)$

Lifting Choose two arbitrary functions ϕ, ψ. In $\mathbb{R}^2 \times \mathbb{R}^2$ we define a pseudoholomorphic curve for the direct product structure by

$$\rho = \phi(x), \quad \tau = \psi(y).$$

On the inverse image of the curve under π we have a rank two integrable distribution.

Integration yields the general solution of the equation:

$$z(x,y) = A(y) + B(x) + B'(x)(y-x).$$

$$p = B'(x) + B''(x)(y-x) - B'(x) = B''(x)(y-x)$$
$$s = B''(x)$$

Sophus Lie

Sophus Lie (1842–1988)

- Geometric structures, partial differential equations, Lie groups
- Symmetries of partial differential equations

Example: CMC surfaces

Consider for example the equation for constant mean curvature in local coordinates.

$$H = \frac{(1 + h_v^2 h_{uu} - 2h_u h_v h_{uv} + (1 + h_u^2)h_{vv}}{2(1 + h_u^2 + h_v^2)^{3/2}} = \text{constant}$$

The equations are elliptic and hence on $\mathcal{V}$ we have an almost complex structure. We can take the quotient of the equation manifold by the translation symmetries.
Let $B = M/\sim$. Then $\pi : M \to B$ defines a a bijective map $T_m\pi\mathcal{V}_m \to T_\pi(m)B$ except at singular points. The complex structure on $\mathcal{V}$ is projected to an almost complex structure J^B on B in the sense that

$$T\pi \circ J = J^B \circ T\pi. \tag{1}$$

CMC equation: solutions

We can lift pseudoholomorphic curves to solutions of the CMC with the same construction as the method of Darboux.

Lifted solutions: Weierstrass representation.

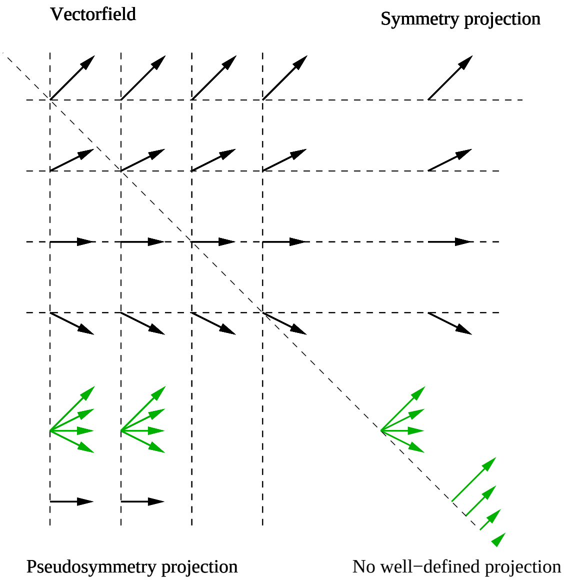
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Re \int \begin{pmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{pmatrix} \mathrm{d}w$$

Generalized Darboux integrability

We have introduced two methods of integrating partial differential equations: the method of Darboux and symmetry reductions (CMC surface). For both methods there is a projection $M \to B$ transversal to $\mathcal{V}$ such that the Monge systems project to B.

	Symmetry reduction	No symmetry
Darboux integrable	Minimal surface equation	$s = p/(y - x)$
Not Darboux integrable	CMC surfaces	?

Pseudosymmetries for vector fields



Pseudosymmetries of distributions

Definition

A pseudosymmetry V of a distribution $\mathcal{V}$ is an integrable distribution $\mathcal{U}$ such that $[\mathcal{U}, \mathcal{V}] \subset \mathrm{span}(\mathcal{V}, \mathcal{U})$.

Locally let B to be the quotient manifold of M by the leaves of $\mathcal{U}$. Then the distribution $\mathcal{V}$ projects to a distribution on B. We have $T_m \pi(\mathcal{V}_m) = \mathcal{W}_{\pi(m)} \subset T_{\pi(m)} B$.

Theorem

Let $(M, \mathcal{V})$ be an elliptic (hyperbolic) Vessiot system. Let $\mathcal{U}$ be a rank 3 distribution on M that is

- *The distribution $\mathcal{U}$ is integrable*
- *The distribution $\mathcal{U}$ is a pseudosymmetry of both Monge systems*

Then locally the leaves of $\mathcal{U}$ define a 4-dimensional manifold B and the projection $\pi : M \to B$ intertwines the complex structure (direct product structure) on $\mathcal{V}$ with an almost complex structure (almost product structure) J^B on B.

The integral manifolds of $\mathcal{V}$ are in one-to-one correspondence (up to an integration constant) with the pseudoholomorphic curves for J^B.

Some results

- Every Darboux integrable equation has a pseudosymmetry.
- Every symmetry reduction by a 3-dimensional symmetry group is a pseudosymmetry.
- Very few projections are both a Darboux projection and a symmetry projection.
- There exists examples of pseudosymmetries that are neither a symmetry reduction nor a Darboux projection.

Messages to take home

- Contact structures can be used to give a geometric description of partial differential equations
- Whenever symmetries are used, maybe pseudosymmetries can be used as well (for making projections, or maybe something completely different)

More: *Contact Structures of Partial Differential Equations*, http://www.math.uu.nl/people/eendebak/